HOMEWORK #6 (DUE FRIDAY, NOV. 21).

11/13/2014

Note: *Turn in only the "starred" problems; out of these, selected problems will be graded.*

Section 7.1.: Exercises 1, 2, 3, 4, 5, 6, 13 14*, 15* 21, 22, 29, 30.

Section 7.2.: Exercises 4, 5, 6, 7*, 8, 12.

Section 7.2.: Exercises 1, 2, 3, 4, 5, 6, 9, 10, 20, 21, 25*, 33*.

Additional problems:

All commutative rings are assumed to have unit and $1 \neq 0$.

1*. Let A be a commutative ring.

(a) Show that a formal power series $f = \sum_{i=0}^{\infty} a_i X^i \in A[[X]]$ is invertible if and only if a_0 is invertible in A.

(b) Show that a polynomial $f = \sum_{i=0}^{n} a_i X^i \in A[X]$ is invertible if and only if a_0 is invertible in A and $a_1, \dots, a_n$ are all nilpotent. (An element a in a commutative ring A is nilpotent if $a^m = 0$ for some $m \geq 1$.)

(c) Show that a polynomial $f = \sum_{i=0}^{n} a_i X^i$ is nilpotent in $A[X]$ if and only if $a_0, a_1, \dots, a_n$ are all nilpotent in A.

2*. Let $(G, +)$ be a finite abelian group whose order is not divisible by the square of any integer. (We say the order is *square-free*.)

(a) Show that, up to isomorphism, there is a unique structure of ring with 1 on G. (That is, show that one can define multiplications on the set G, which together with the given additive operation form structures of ring with 1, then show that all these rings are isomorphic.) Observe that this unique ring structure is commutative.

(b) Let A denote the ring $(G, +, \cdot)$ from part (a) above. Prove that the multiplicative group $A[X]^*$ of invertible polynomials in $A[X]$ coincides with the multiplicative group A^* of invertible elements in A. (Exercise 1(b) is relevant here; Exercise 4 might help, but you don't really need it.)

(c) Show by examples that both (a) and (b) are false if we don't assume that the order of G is not divisible by a square.

3*. Let d be a square-free nonzero integer.

(i) Show that

$$\mathbb{Z}[\sqrt{d}] := \{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}$$

is an integral domain, isomorphic to $\mathbb{Z}[X]/(X^2 - d)$.

(ii) If $x = a + b\sqrt{d}$, find a generator t of the ideal $(x) \cap \mathbb{Z}$, and a generator r of the ideal $\{m \in \mathbb{Z} \mid m\sqrt{d} \in (x)\}$.

(iii) Is it true in general that $(x) = t\mathbb{Z} + r\sqrt{d}\mathbb{Z}$?

4*. Let A, B be commutative rings and let $\mathfrak{N}(A)$ (respectively, $\mathfrak{N}(B)$) denote the set of nilpotent elements in A (respectively, in B).

(i) Show that $\mathfrak{N}(A)$ is an ideal.

(ii) Show that $\mathfrak{N}(A \times B) = \mathfrak{N}(A) \times \mathfrak{N}(B)$.