HOMEWORK #7 (DUE WEDNESDAY, DEC. 3).

11/25/2014

Note: *Turn in only the "starred" problems; out of these, selected problems will be graded.*

Section 7.4: Exercises 6, 7, 9,10, 11*, 12, 13*, 15, 19, 36, 37.

Section 7.5.: Exercises 5, 6.

Section 7.6.: Exercises 3*, 5*. (For Exercise 5 use the Chinese Remainder Theorem as stated in Theorem 17, page 265 of the textbook; note that in class we proved the version stating that congruences modulo pairwise coprime ideals can be solved simultaneously and derived from it the statement of Thm. 17, so part (a) of this exercise shows the converse implication, at least for $\mathbb{Z}$.)

Additional problems:

1*. Describe all the prime ideals in $\mathbb{Z}[X]$.

In the rest of the problems all commutative rings have 1 and all multiplicative systems contain 1, as defined in class. However, unless otherwise stated, multiplicative systems may contain zero divisors.

2. Let A be a commutative ring and let $S \subset A$ be a multiplicative system which does not contain zero divisors. Show that the canonical inclusion $S^{-1}(A[X_1, \dots, X_n]) \subset (S^{-1}A)[X_1, \dots, X_n]$ is an isomorphism, but in general the inclusion $S^{-1}(A[[X]]) \subset (S^{-1}A)[[X]]$ is not bijective.

3*. Let A be a commutative ring and let $f \in A$ be an element which is not a zero divisor. Show that $A_f \cong A[X]/(fX - 1)$. (Recall that A_f denotes the ring of fractions $S^{-1}A$ for $S = \{1, f, f^2, \dots\}$.)

4*. Let $\phi : A \subset B$ be a morphism of commutative rings, S a multiplicative system in A, and T a multiplicative system in B with $\phi(S) \subset T$. Show that there exists an unique morphism $\phi' : S^{-1}A \to T^{-1}B$ such that $\phi i_T = i_S \phi'$, where $i_S : A \to S^{-1}A$ and $i_T : B \to T^{-1}B$ are the canonical morphisms.

5*. Let A be a commutative ring and let $S \subset A$ be a multiplicative system which does not contain zero divisors. Put

$$T = \{x \in A \mid \text{there are } s \in S, y \in A \text{ with } s = xy\}.$$

(i) Show that T is a multiplicative system with the property that $xy \in T$ iff $x \in T$ and $y \in T$. (In general a multiplicative system with this property is called *saturated*. The system T defined above is called the *saturation of* S.)

(ii) Show that $S^{-1}A \cong T^{-1}A$.