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ON QUANTUM COHOMOLOGY RINGS OF PARTIAL FLAG VARIETIES

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0. Introduction. The main goal of this paper is to give a unified description for the structure of the small quantum cohomology rings for all projective homogeneous spaces $SL_n(\mathbb{C})/P$, where P is a parabolic subgroup.

The quantum cohomology ring of a smooth projective variety, or more generally of a symplectic manifold X, has been introduced by string theorists (see [Va] and [W1]). Roughly speaking, it is a deformation of the usual cohomology ring, with parameter space given by $H^*(X)$. The multiplicative structure of quantum cohomology encodes the enumerative geometry of rational curves on X. In the past few years, the highly nontrivial task of giving a rigorous mathematical treatment of quantum cohomology has been accomplished both in the realm of algebraic and symplectic geometry. In various degrees of generality, this can be found in [Beh], [BehM], [KM], [LiT1], [LiT2], [McS], and [RT], as well as in the surveys [FP] and [T].

If one restricts the parameter space to $H^{1,1}(X)$, one gets the *small quantum cohomology ring*. This ring, in the case of partial flag varieties, is the object of the present paper. In order to state our main results, we first describe briefly the "classical" side of the story.

We interpret the homogeneous space $F := SL_n(\mathbb{C})/P$ as the complex, projective variety, parametrizing flags of *quotients* of $\mathbb{C}^n$ of given ranks, say, $n_k > \cdots > n_1$.

By a classical result of Ehresmann [E], the integral cohomology of F can be described geometrically as the free abelian group generated by the *Schubert classes*. These are the (Poincaré duals of) fundamental classes of certain subvarieties $\Omega_w \subset F$, one for each element of the subset $S := S(n_1, \ldots, n_k)$ of the symmetric group S_n, consisting of permutations w with descents in $\{n_1, \ldots, n_k\}$.

A description of the multiplicative structure is provided by yet another classical theorem, due to Borel [Bor], which gives a presentation for $H^*(F, \mathbb{Z})$. Specifically, let $\sigma_1^1, \ldots, \sigma_{n_1}^1, \sigma_1^2, \ldots, \sigma_{n_2-n_1}^2, \ldots, \sigma_1^{k+1}, \ldots, \sigma_{n-n_k}^{k+1}$ be n independent variables. Define A_n to be the block-diagonal matrix $\operatorname{diag}(D_1, D_2, \ldots, D_{k+1})$, where

$$D_j := \begin{pmatrix} \sigma_1^j & \sigma_2^j & \cdots & \sigma_{n_j-n_{j-1}-1}^j & \sigma_{n_j-n_{j-1}}^j \\ -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

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Borel's result states that there is a canonical isomorphism

$$\mathbb{Z}\Big[\sigma_1^1,\dots,\sigma_{n_1}^1,\dots,\sigma_1^{k+1},\dots,\sigma_{n-n_k}^{k+1}\Big]\Big/(g_1,g_2,\dots,g_n)\cong H^*(F), \tag{0.1}$$

with $g_1,\dots,g_n$ the coefficients of the characteristic polynomial of the matrix A_n.

The natural problem arising from the above descriptions is to look for polynomial representatives for the Schubert classes. The first case in which this problem was solved is when F is a Grassmannian, which goes back to Schur and Giambelli.

The general case was obtained independently by Bernstein, Gelfand, and Gelfand [BGG] and by Demazure [D]. In fact, it suffices to solve the above problem for the *complete* flag variety $F_n = SL_n(\mathbb{C})/B$. The point is that the map

$$H^*(F,\mathbb{Z}) \longrightarrow H^*(F_n,\mathbb{Z}) \tag{0.2}$$

induced by flat pullback via the natural projection $F_n \to F$ is an embedding. More precisely, the Borel description for the cohomology of the complete flag variety is

$$\mathbb{Z}[x_1,x_2,\dots,x_n]/(e_1,\dots,e_n)\cong H^*(F_n,\mathbb{Z}),$$

where e_j is the jth elementary symmetric polynomial in $x_1,\dots,x_n$. A particularly nice set of representatives for the Schubert classes in this case is given by the *Schubert polynomials* $\mathfrak{S}_w(x_1,\dots,x_n)$ of Lascoux and Schützenberger [LS]. If we interpret each σ_i^j as the ith elementary symmetric polynomial in variables $x_{n_{j-1}+1},\dots,x_{n_j}$, then the image of $H^*(F,\mathbb{Z})$ by the map (0.2) is the subring of polynomials that are symmetric in variables in each of the groups

$$\underbrace{x_1,\dots,x_{n_1}}, \underbrace{x_{n_1+1},\dots,x_{n_2}},\dots,\underbrace{x_{n_k+1},\dots,x_n}.$$

If $w \in S$, then $\mathfrak{S}_w$ satisfies the above symmetry; hence, it determines a polynomial P_w in the σ variables, which represents $[\Omega_w]$ in $H^*(F,\mathbb{Z})$. We call the $P_w(\sigma)$'s the *Giambelli polynomials* associated to F. (When F is a Grassmannian, these are the Giambelli determinants in the Chern classes of the universal quotient.)

Within the Schubert varieties are the *special Schubert varieties*, which are geometric realizations of the Chern classes of the universal quotient bundles on F. They correspond to the cyclic permutations $\alpha_{i,j} := s_{n_j-i+1}\times\cdots\times s_{n_j}$, for $1\le j\le k$ and $1\le i\le n_j$, where $s_m := (m,m+1)$ is the simple transposition interchanging m and $m+1$. Again, when F is a Grassmannian, there is a classical formula, due to Pieri, expressing the product $[\Omega_{\alpha_{i,1}}]\cdot[\Omega_w]$ in the basis of Schubert classes. Its generalization to the case of the complete flag variety, and hence, by the above discussion, to any partial flag variety as well, was first stated by Lascoux and Schützenberger [LS] and was given a geometric proof by Sottile [So].

In analogy to the case of Grassmannians, we refer to the Giambelli- and Pieri-type formulas as the *classical Schubert calculus* on F.

The small quantum cohomology ring of F, denoted by $QH^*(F)$, is defined as the $\mathbb{Z}[q_1,\dots,q_k]$-module $H^*(F,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[q_1,\dots,q_k]$, where $q_1,\dots,q_k$ are formal variables, with a new multiplication, which we denote by $*$. This multiplication is obtained by replacing the classical structure constants with polynomials in $q_1,\dots,q_k$, whose coefficients are the *3-point, genus-zero Gromov-Witten (GW) invariants* of F. A presentation of $QH^*(F)$ was given independently by Astashkevich and Sadov [AS] and by Kim [Kim1], with the proof completed in [Kim2]. (The "extreme" cases of Grassmannians and complete flags were established slightly earlier in [ST], [W2] and [CF1], [GK], respectively.) Their result is as follows. Let $B_n=(b_{lm})_{1\le l,m\le n}$ be the matrix with entries

$$b_{lm}=\begin{cases}(-1)^{n_{j+1}-n_j+1}q_j, & \text{if } l=n_{j-1}+1 \text{ and } m=n_{j+1},\ \text{for } 1\le j\le k,\\ -1, & \text{if } l=n_j+1 \text{ and } m=n_j,\ \text{for } 1\le j\le k-1,\\ 0, & \text{otherwise.}\end{cases}$$

Then there exists a canonical isomorphism

$$\mathbb{Z}\Big[\sigma_1^1,\dots,\sigma_{n_1}^1,\dots,\sigma_1^{k+1},\dots,\sigma_{n-n_k}^{k+1}\Big][q_1,\dots,q_k]\big/\big(G_1,\dots,G_n\big)\cong QH^*(F), \tag{0.3}$$

where $G_1,\dots,G_n$ are the coefficients of the characteristic polynomial of the deformed matrix $A_n^q:=A_n+B_n$.

From the point of view of enumerative geometry, one is interested in computing the Gromov-Witten invariants of F, and the description (0.3) is not too helpful unless one has quantum versions of the Giambelli and Pieri formulas. In other words, one is interested in developing a *quantum Schubert calculus*. The first such formulas, in the case where F is a Grassmannian, were discovered by Bertram, whose paper [Be] started the subject. Later, his approach was extended to the case of complete flags to obtain the quantum Giambelli formula for the *special* Schubert classes (see [CF1] and [CF2]). Using this, the *quantum Schubert polynomials* were constructed with algebro-combinatorial methods by Fomin, Gelfand, and Postnikov [FoGP], therefore giving the full quantum Giambelli formula for the variety of complete flags. They also gave a special case of the quantum Pieri formula, namely, the quantum Monk formula, which corresponds to multiplying by the *first* Chern class of one of the tautological bundles.

As opposed to the situation of the classical cohomology, the quantum story for a partial flag variety is far from being determined by the one for the complete flags. The reason for this is that the quantum cohomology lacks the functoriality enjoyed by the usual one. The main results of this paper are general quantum versions of the Giambelli and Pieri formulas, which hold for *any* F. These formulas specialize to the ones mentioned above when F is either a Grassmannian or the complete flag variety. In order to state them, we first introduce some notation.

Let $1 \le h_1 < \cdots < h_m \le l_m < \cdots < l_1 \le k$ be integers. We denote by $\mathbf{h}$ and $\mathbf{l}$, the collections $h_1, \ldots, h_m$ and $l_1, \ldots, l_m$, respectively. Set

$$\gamma_{\mathbf{hl}} := \gamma_{h_m,l_m} \cdot \gamma_{h_{m-1},l_{m-1}} \times \cdots \times \gamma_{h_1,l_1}, \qquad \delta_{\mathbf{hl}} := \delta_{h_1,l_1} \cdot \delta_{h_2,l_2} \times \cdots \times \delta_{h_m,l_m},$$

where $\gamma_{h,l}$ and $\delta_{h,l}$ are the cyclic permutations $s_{n_h} \times \cdots \times s_{n_{l+1}-1}$ and $s_{n_l-1} \times \cdots \times s_{n_{h-1}+1}$, respectively, for any integers $1 \le h \le l \le k$. Denote by $q_{\mathbf{hl}}$ the monomial

$$q_{\mathbf{hl}} := \underbrace{q_{h_1} \cdots q_{h_2-1}}\underbrace{q_{h_2}^2 \cdots q_{h_3-1}^2} \cdots \underbrace{q_{h_m}^m \cdots q_{l_m}^m}\underbrace{q_{l_m-1}^{m-1} \cdots q_{l_{m-1}}^{m-1}} \cdots \underbrace{q_{l_2-1} \cdots q_{l_1}}.$$

For each $1 \le j \le k$ and $1 \le i \le n_j$, let $\alpha_{i,j} = s_{n_j-i+1} \times \cdots \times s_{n_j}$. For $1 \le a < b \le n$, denote by t_{ab} the transposition interchanging a and b. If $w, w' \in S$, write $w \xrightarrow{\alpha_{i,j}} w'$ if there exist integers $a_1, b_1, \ldots, a_i, b_i$, such that

(1) $a_r \le n_j < b_r$, for $1 \le r \le i$, and $w' = w \cdot t_{a_1b_1} \times \cdots \times t_{a_ib_i}$;
(2) $\ell(w \cdot t_{a_1b_1} \times \cdots \times t_{a_rb_r}) = \ell(w) + r$, for $1 \le r \le i$;
(3) the integers $a_1, \ldots, a_i$ are *distinct*.

Our first main theorem is the following.

QUANTUM PIERI FORMULA. *For $1 \le j \le k$, $1 \le i \le n_j$, and $w \in S$,*

$$\left[\Omega_{\alpha_{i,j}}\right] * [\Omega_w] = \sum_{w \xrightarrow{\alpha_{i,j}} w'} \left[\Omega_{w'}\right] + \sum_{\mathbf{h},\mathbf{l}} q_{\mathbf{hl}} \left(\sum_{w''} \left[\Omega_{w'' \cdot \delta_{\mathbf{hl}}}\right] \right),$$

where the second sum is over all collections $\mathbf{h}, \mathbf{l}$ such that $m \le i$, $h_m \le j \le l_m$, and

$$\ell(w \cdot \gamma_{\mathbf{hl}}) = \ell(w) - \sum_{c=1}^{m} \left(n_{l_c+1} - n_{h_c}\right),$$

while the last sum is over all permutations $w'' \in S_n$ satisfying $w \cdot \gamma_{\mathbf{hl}} \xrightarrow{\tilde{\alpha}_{i,j}} w''$, with $\tilde{\alpha}_{i,j} = \alpha_{i,j} \cdot s_{n_j} \cdot s_{n_j-1} \times \cdots \times s_{n_j-m+1}$, and

$$\ell\left(w'' \cdot \delta_{\mathbf{hl}}\right) = \ell\left(w''\right) - m - \sum_{c=1}^{m} \left(n_{l_c} - n_{h_c-1}\right).$$

For each $1 \le j \le k$ and $1 \le i \le n_j$, let $g_i^j = g_i^j(\sigma)$ be the polynomial representing the ith Chern class of the jth universal quotient bundle on F. Alternatively, for each j, the polynomials g_i^j, for $1 \le i \le n_j$, are the coefficients of the characteristic polynomial of the upper-left $n_j \times n_j$ submatrix of the matrix A_n.

Define now polynomials $G_i^j = G_i^j(\sigma, q)$, for $1 \le j \le k$ and $1 \le i \le n_j$, in exactly the same way as above but using the Astashkevich-Sadov-Kim matrix A_n^q instead of

A_n. For a partition $\Lambda_j := (\lambda_{j,1}, \ldots, \lambda_{j,n_{j+1}-n_j})$, with (at most) $n_{j+1} - n_j$ parts, and such that each part $\lambda_{j,m}$ is at most n_j, set

$$g^{(j)}_{\Lambda_j} := g^j_{\lambda_{j,1}} g^j_{\lambda_{j,2}} \cdots g^j_{\lambda_{j,n_{j+1}-n_j}}.$$

Define *elementary monomials* $g_\Lambda \in \mathbb{Z}[\sigma]$ by $g_\Lambda := g^{(1)}_{\Lambda_1} g^{(2)}_{\Lambda_2} \cdots g^{(k)}_{\Lambda_k}$.

The *quantum elementary monomial* G_Λ is the polynomial in $\mathbb{Z}[\sigma, q]$ obtained by replacing in g_Λ each factor g^j_λ by the corresponding G^j_λ. It is easy to see that each Giambelli polynomial can be written uniquely as a linear combination $P_w = \sum_\Lambda a_\Lambda(w) g_\Lambda$, with $a_\Lambda(w)$ integers.

Following [FoGP], define the *quantum Giambelli polynomial* $P^q_w(\sigma, q)$ by

$$P^q_w(\sigma, q) = \sum_\Lambda a_\Lambda(w) G_\Lambda.$$

We then prove the following theorem.

QUANTUM GIAMBELLI FORMULA. *We have $[\Omega_w] = P^q_w(\sigma, q)$ in $QH^*(F)$, for all $w \in S$.*

The proof of the quantum Pieri formula and the main step in the proof of the quantum Giambelli formula are obtained via geometric computations of certain Gromov-Witten invariants of F. This is accomplished by realizing the invariants as intersection numbers on *hyperquot schemes*. For a flag variety F, these schemes provide a compactification of the space of maps from $\mathbb{P}^1$ to F, different than the Kontsevich space of 3-pointed, stable maps. Using degenerations and a detailed description of the boundary of our compactification, we show that the appropriate invariants either vanish or can be expressed in terms of classical intersection numbers on F itself.

This paper is divided into two main parts. The first part starts with a brief review in Sections 1 and 2 of the results about the classical and quantum cohomology rings that we need later. Section 3 is mainly algebra and contains the full description of quantum Schubert calculus. Starting with the quantum Pieri formula (whose proof is deferred for Section 6), we first establish quantum Giambelli for the special Schubert classes, and we also give an independent proof of the Astashkevich-Sadov-Kim theorem. Assuming one more key result (Theorem 3.14), we deduce the general Giambelli formula. We conclude by describing several related results, including a "dual" quantum Pieri formula and another definition of quantum Giambelli polynomials, using divided difference operators (cf. [KiMa], [CFF], and [F3]).

The second part is almost entirely geometric. Section 4 contains a review of the construction of Gromov-Witten invariants via hyperquot schemes (cf. [Be], [CF1], and [CF2]), while Section 5 deals with the geometry of hyperquot schemes. This material is essentially adapted from [CF2], where the case of complete flags is treated. In Section 6 we first discuss a certain degeneration technique, which permits in some cases explicit computations of Gromov-Witten invariants. Finally, we use this technique, together with the results in the previous two sections, to prove quantum Pieri and complete the proof of quantum Giambelli, by establishing the key result mentioned above.

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1. The classical cohomology ring

1.1. Schubert varieties. Let $0=n_0<n_1<n_2<\cdots<n_k<n_{k+1}=n$ be integers. Let V be a complex n-dimensional vector space. The data $k,\ n_j,\ j=0,\dots,k+1$, and V are fixed for the rest of the paper. Define $F:=F(n_1,\dots,n_k,V)$ to be the variety parametrizing flags of *quotients* of V, with ranks given by the n_j's. F is a smooth, irreducible, projective variety, of dimension $f:=\sum_{j=1}^{k}(n-n_j)(n_j-n_{j-1})$. It comes with a tautological sequence of quotient bundles

$$V_F:=V\otimes\mathbb{O}_F\twoheadrightarrow Q_k\twoheadrightarrow Q_{k-1}\twoheadrightarrow\cdots\twoheadrightarrow Q_1,$$

with $\operatorname{rank}(Q_j)=n_j$.

Let S_n be the symmetric group on n letters, and let $S:=S(n_1,\dots,n_k)\subset S_n$ be the subset consisting of permutations w with descents in $\{n_1,\dots,n_k\}$. In other words, when regarded as a function $[1,n]\to[1,n]$, w is increasing on each of the intervals $[1,n_1],[n_1+1,n_2],\dots,[n_k+1,n_{k+1}]$. The *rank function* of a permutation $w\in S_n$ is defined by

$$r_w(q,p)=\operatorname{card}\big\{i\mid i\le q,w(i)\le p\big\},\qquad \text{for } 1\le q,p\le n.$$

Fix a complete flag of subspaces $V_1\subset V_2\subset\cdots\subset V_{n-1}\subset V_n=V$. For $w\in S$, the corresponding Schubert variety is defined by

$$\Omega_w:=\Big\{x\in F\mid \operatorname{rank}_x\big(V_p\otimes\mathbb{O}\to Q_q\big)\le r_w(q,p),q\in\{n_1,\dots,n_k\},1\le p\le n\Big\}.$$

Ω_w is an irreducible subvariety in F of (complex) codimension equal to the length $\ell(w)$ of the permutation w. The *dual Schubert variety* to Ω_w is the Schubert variety corresponding to the permutation $\check{w}\in S$, given by

$$\check{w}(i)=n-w\big(n_j+n_{j-1}-i+1\big)+1,\quad \text{for } n_{j-1}+1\le i\le n_j,1\le j\le k.$$

Throughout this paper, $H^*(F)$ denotes the integral cohomology of F. The following two theorems are classical results of Ehresmann [E] (see also [F2, Example 14.7.16]).

THEOREM 1.1 (Basis). *$\{[\Omega_w]\}_{w\in S}$ freely generate $H^*(F)$ over $\mathbb{Z}$.*

THEOREM 1.2 (Duality). *For every $w \in S$,*

$$\int_F [\Omega_w] \cup [\Omega_v] = \begin{cases} 1, & \text{if } v = \check{w}, \\ 0, & \text{otherwise.} \end{cases}$$

1.2. A presentation of $H^*(F)$. Consider on F the vector bundles

$$L_j := \ker\big(Q_j \to Q_{j-1}\big),$$

and let $\sigma_i^j := c_i(L_j)$, for $1 \le i \le n_j - n_{j-1}$ and $1 \le j \le k+1$. Let $x_1, \dots, x_n$ be independent variables. For all $0 \le i \le m \le n$, let e_i^m denote the ith elementary symmetric function in the variables $x_1, \dots, x_m$. We regard the variables in each of the groups

$$\underbrace{x_1, \dots, x_{n_1}}, \underbrace{x_{n_1+1}, \dots, x_{n_2}}, \dots, \underbrace{x_{n_k+1}, \dots, x_n}$$

as the Chern roots of the bundles Q_1 and $L_2, \dots, L_{k+1}$, respectively. For each $1 \le j \le k+1$, the polynomials $e_i^{n_j}$ can be written as polynomials $g_i^j = g_i^j(\sigma_1^1, \dots, \sigma_{n_{k+1}-n_k}^{k+1})$ in the Chern classes of these bundles. The polynomial g_i^j has weighted degree i, where each σ_m^* is assigned degree m. In particular, we have polynomials g_i^{k+1}, for $1 \le i \le n$.

Denote the polynomial ring $\mathbb{Z}[\sigma_1^1, \dots, \sigma_{n_1}^1, \sigma_1^2, \dots, \sigma_{n_2-n_1}^2, \dots, \sigma_1^{k+1}, \dots, \sigma_{n-n_k}^{k+1}]$ by $\mathbb{Z}[\sigma]$. The following is another classical result, due to Borel [Bor].

THEOREM 1.3. *$H^*(F)$ is canonically isomorphic to $\mathbb{Z}[\sigma]/(g_1^{k+1}, \dots, g_n^{k+1})$.*

1.3. Classical Schubert calculus for F. Let us recall the Schubert polynomials of Lascoux and Schützenberger [LS]. Define operators ∂_i, for $i = 1, \dots, n-1$, on $\mathbb{Z}[x_1, \dots, x_n]$ by

$$\partial_i P = \frac{P(x_1, \dots, x_n) - P(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n)}{x_i - x_{i+1}}.$$

For any $w \in S_n$, write $w = w_\circ \cdot s_{i_1} \times \cdots \times s_{i_k}$, with $k = n(n-1)/2 - \ell(w)$, where $s_i = (i, i+1)$ is the transposition interchanging i and $i+1$, and where $w_\circ$ is the permutation of longest length, given by $w_\circ(j) = n - j + 1$, for $1 \le j \le n$. The *Schubert polynomial* $\mathfrak{S}_w(x) \in \mathbb{Z}[x_1, \dots, x_n]$ is defined by

$$\mathfrak{S}_w(x) = \partial_{i_k} \circ \cdots \circ \partial_{i_1}\big(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}\big).$$

It is shown in [M] that if $w \in S$, then the corresponding Schubert polynomial is symmetric in each group of variables

$$\underbrace{x_1, \dots, x_{n_1}}, \underbrace{x_{n_1+1}, \dots, x_{n_2}}, \dots, \underbrace{x_{n_k+1}, \dots, x_n};$$

hence, it can be written as a polynomial $P_w(\sigma)$ of weighted degree $\ell(w)$. We call these $P_w(\sigma)$ *Giambelli polynomials*. The following theorem is due to Bernstein, Gelfand, and Gelfand [BGG] and Demazure [D].

THEOREM 1.4 (Giambelli-type formula). $[\Omega_w] = P_w(\sigma)$ *in* $H^*(F)$.

In particular, consider the cyclic permutations (of length i) $\alpha_{i,j} := s_{n_j-i+1} \times \cdots \times s_{n_j}$ and $\beta_{i,j} := s_{n_j+i-1} \times \cdots \times s_{n_j}$. Note that these permutations are in S. Their Schubert polynomials are $\mathfrak{S}_{\alpha_{i,j}} = e_i^{n_j}$ and $\mathfrak{S}_{\beta_{i,j}} = h_i^{n_j}$, where $h_i^{n_j}$ is the ith *complete* symmetric polynomial in variables $x_1, \dots, x_{n_j}$. Let $f_i^j(\sigma)$ be the polynomial in the σ-variables obtained from $h_i^{n_j}$. By Theorem 1.4,

$$[\Omega_{\alpha_{i,j}}] = g_i^j \quad \text{and} \quad [\Omega_{\beta_{i,j}}] = f_i^j \quad \text{in } H^*(F). \tag{1.1}$$

The following Pieri-type formula, due to Lascoux and Schützenberger [LS], recently was given a geometric proof by Sottile [So].

Let $w, w' \in S$. For $1 \le a < b \le n$, denote by t_{ab} the transposition interchanging a and b. Write $w \xrightarrow{\alpha_{i,j}} w'$ if there exist integers $a_1, b_1, \dots, a_i, b_i$, satisfying

(1) $a_m \le n_j < b_m$, for $1 \le m \le i$, and $w' = w \cdot t_{a_1 b_1} \times \cdots \times t_{a_i b_i}$;
(2) $\ell(w \cdot t_{a_1 b_1} \times \cdots \times t_{a_m b_m}) = \ell(w) + m$, for $1 \le m \le i$; and
(3α) the integers $a_1, \dots, a_i$ are *distinct*.

Similarly, $w \xrightarrow{\beta_{i,j}} w'$ if there exist $a_1, b_1, \dots, a_i, b_i$ as above, satisfying (1), (2) and
(3β) $b_1, \dots, b_i$ are *distinct*.

THEOREM 1.5 (Pieri formula). *The following hold in* $H^*(F)$*:*
(i) $[\Omega_{\alpha_{i,j}}] \cdot [\Omega_w] = \sum_{w \xrightarrow{\alpha_{i,j}} w'} [\Omega_{w'}]$;
(ii) $[\Omega_{\beta_{i,j}}] \cdot [\Omega_w] = \sum_{w \xrightarrow{\beta_{i,j}} w'} [\Omega_{w'}]$.

Remark 1.6. From the exact sequence

$$0 \longrightarrow L_j \longrightarrow Q_j \longrightarrow Q_{j-1} \longrightarrow 0,$$

we get $c_i(Q_j) = \sum_{r=0}^{n_j - n_{j-1}} \sigma_r^j c_{i-r}(Q_{j-1})$. But one easily sees that $c_i(Q_j) = [\Omega_{\alpha_{i,j}}]$. Using (1.1), it follows that the polynomials g_i^j satisfy the following recursion (which in fact defines them uniquely):

$$g_i^j = \sum_{r=0}^{n_j - n_{j-1}} \sigma_r^j g_{i-r}^{j-1}, \tag{1.2}$$

where, by convention, we set $g_0^{j-1} = 1$ and $g_m^{j-1} = 0$, if either $m < 0$ or $m > n_{j-1}$.

Also, using the same exact sequence, the relations (1.1) and (1.2), and the well-known identity

$$\left(\sum_{r=0}^{n_{j-1}} e_r^{n_{j-1}} t^r\right)^{-1} = \sum_{p\geq 0}(-1)^p h_p^{n_{j-1}} t^p,$$

we get that the following identities hold in $H^*(F)$:

$$\sigma_r^j = \sum_{p=0}^{r}(-1)^p\big[\Omega_{\beta_{p,j-1}}\big]\cdot\big[\Omega_{\alpha_{r-p,j}}\big], \tag{1.3}$$

$$\big[\Omega_{\alpha_{i,j}}\big] = \sum_{r=0}^{n_j-n_{j-1}}\left(\sum_{p=0}^{r}(-1)^p\big[\Omega_{\beta_{p,j-1}}\big]\cdot\big[\Omega_{\alpha_{r-p,j}}\big]\right)\cdot\big[\Omega_{\alpha_{i-r,j-1}}\big], \tag{1.4}$$

where, by convention, $[\Omega_{\alpha_{0,m}}] = [\Omega_{\beta_{0,m}}] = 1$ and $[\Omega_{\alpha_{<0,m}}] = 0$, for all m.

2. The small quantum cohomology ring of F. We give below the precise definition of the small quantum cohomology ring only for the specific case of a partial flag manifold.

The *3-point, genus-zero Gromov-Witten (GW) invariants* of F, which we denote by $I_{3,\beta}(\gamma_1\gamma_2\gamma_3)$, are defined as intersection numbers on Kontsevich's moduli space of stable maps $\overline{M}_{0,3}(F,\beta)$ (see [KM], [K], [BehM], and [FP]). Here $\beta \in H_2(F)$ and $\gamma_1,\gamma_2,\gamma_3 \in H^*(F)$. The enumerative significance of these numbers is given by the following result, whose proof can be found in [FP, Lemma 14].

LEMMA 2.1. *Let Γ_1,Γ_2, and Γ_3 be closed subvarieties of F representing the cohomology classes γ_1,γ_2, and γ_3, respectively. Let $g_1,g_2,g_3 \in SL(n,\mathbb{C})$ be general elements, and denote by $g_i\Gamma_i$ the translate of Γ_i by g_i. Then $I_{3,\beta}(\gamma_1\gamma_2\gamma_3)$ is the number of maps $\mu : \mathbb{P}^1 \to F$ such that $\mu_*[\mathbb{P}^1] = \beta$ and $\mu(\mathbb{P}^1)$ meets $g_1\Gamma_1, g_2\Gamma_2$, and $g_3\Gamma_3$.*

Since we give a different construction of these invariants in Section 4, we shall say no more about them here. The multiplication in the (small) quantum cohomology ring is defined using these $I_{3,\beta}$ as structure constants. More precisely, this goes as follows.

Introduce formal variables $q_1,\dots,q_k$, corresponding, respectively, to the generators (cf. Theorem 1.1) $[\Omega_{\check{s}_{n_1}}],\dots,[\Omega_{\check{s}_{n_k}}]$ of $H_2(F)$. For a (holomorphic) map $\mu : \mathbb{P}^1 \to F$, we can write $\beta = \mu_*[\mathbb{P}^1] = \sum_{j=1}^{k} d_j[\Omega_{\check{s}_{n_j}}]$, with d_j nonnegative integers. We say that μ has multidegree $\overline{d} = (d_1,\dots,d_k)$, and we replace β by $\overline{d}$ in the notation for GW invariants.

Let $K := \mathbb{Z}[q_1,\dots,q_k]$. On the K-module $H^*(F)\otimes_{\mathbb{Z}} K$, define the quantum multiplication $*$ by putting first

$$\big[\Omega_u\big]*\big[\Omega_v\big] := \sum_{\overline{d}} q_1^{d_1}\cdots q_k^{d_k}\sum_{w\in S} I_{3,\overline{d}}\big(\big[\Omega_u\big]\big[\Omega_v\big]\big[\Omega_w\big]\big)\big[\Omega_{\check{w}}\big], \tag{2.1}$$

for all $u,v \in S$, and then extending linearly on $H^*(F)$ and trivially on K. The following theorem is a particular case of the general results on associativity of quantum cohomology (see [Beh], [BehM], [KM], [LiT1], [LiT2], [McS], and [RT]).

THEOREM 2.2. *The operation $*$ defines an associative and commutative K-algebra structure on $H^*(F)\otimes_{\mathbb{Z}} K$.*

$H^*(F)\otimes_{\mathbb{Z}} K$ together with this multiplication is called the small quantum cohomology ring of F and is denoted by $QH^*(F)$. The goal of this paper is to give a description analogous to that in Section 1 for this new algebra.

3. Quantum Schubert calculus

3.1. The quantum version of the Pieri formula. We first introduce some notation. For integers h,l satisfying $1\le h\le l\le k$, consider the cyclic permutations $\gamma_{h,l}:= s_{n_h}\times\cdots\times s_{n_{l+1}-1}$ and $\delta_{h,l}:=s_{n_l-1}\times\cdots\times s_{n_{h-1}+1}$. Now let $1\le j\le k$ and $1\le i\le n_j$ be fixed, and let

$$m\le i,\qquad 1\le h_1<\cdots<h_m\le j\le l_m<\cdots<l_1\le k$$

be integers. We denote by $\mathbf{h}$ and $\mathbf{l}$ the collections $h_1,\dots,h_m$ and $l_1,\dots,l_m$, respectively. Let

$$\gamma_{\mathbf{hl}}:=\gamma_{h_m,l_m}\cdot\gamma_{h_{m-1},l_{m-1}}\times\cdots\times\gamma_{h_1,l_1};\qquad \delta_{\mathbf{hl}}:=\delta_{h_1,l_1}\cdot\delta_{h_2,l_2}\times\cdots\times\delta_{h_m,l_m};$$

$$q_{\mathbf{hl}}:=q_{h_1}\cdots q_{h_2-1}q_{h_2}^2\cdots q_{h_3-1}^2\cdots q_{h_m}^m\cdots q_{l_m}^m q_{l_m-1}^{m-1}\cdots q_{l_{m-1}}^{m-1}\cdots q_{l_2-1}\cdots q_{l_1}.$$

THEOREM 3.1 (Quantum Pieri formula). *For $1\le j\le k$, $1\le i\le n_j$, and $w\in S$,*

$$[\Omega_{\alpha_{i,j}}]*[\Omega_w]=\sum_{w\xrightarrow{\alpha_{i,j}}w'}[\Omega_{w'}]+\sum_{\mathbf{h},\mathbf{l}}q_{\mathbf{hl}}\left(\sum_{w''}[\Omega_{w''\cdot\delta_{\mathbf{hl}}}]\right),\tag{3.1}$$

where the second sum is over all collections $\mathbf{h},\mathbf{l}$ such that

$$\ell(w\cdot\gamma_{\mathbf{hl}})=\ell(w)-\sum_{c=1}^{m}(n_{l_c+1}-n_{h_c}),$$

while the last sum is over all permutations $w''\in S_n$ satisfying $w\cdot\gamma_{\mathbf{hl}}\xrightarrow{\tilde{\alpha}_{i,j}}w''$, with $\tilde{\alpha}_{i,j}=\alpha_{i,j}\cdot s_{n_j}\cdot s_{n_j-1}\times\cdots\times s_{n_j-m+1}$ and

$$\ell(w''\cdot\delta_{\mathbf{hl}})=\ell(w'')-m-\sum_{c=1}^{m}(n_{l_c}-n_{h_c-1}).$$

Remark 3.2. (i) The first term in the right-hand side of the formula is the "classical" one, given by Theorem 1.5.

(ii) The condition $\ell(w\cdot\gamma_{\mathbf{hl}}) = \ell(w) - \sum_{c=1}^{m}(n_{l_c+1} - n_{h_c})$ can be rephrased equivalently as

$$(*) \qquad w\big(n_{h_c}\big) > \max\big\{w(n_{h_c}+1),\dots,w(n_{l_c+1})\big\}, \qquad \text{for } 1 \le c \le m.$$

(iii) Note that if $m < i$, then $\tilde{\alpha}_{i,j} = s_{n_j-i+1} \times \cdots \times s_{n_j-m}$ gives the same kind of cyclic permutation as $\alpha_{i,j}$, but it determines a Schubert variety only on the flag varieties for which one of the quotients has rank $n_j - m$! In fact, as seen in the proof of Theorem 3.1, the last sum in the formula comes from applying Theorem 1.5 on such a flag variety. That is, the permutations w'' are obtained from the terms in the "classical Pieri" for the multiplication of Schubert varieties corresponding to $w\cdot\gamma_{\mathbf{hl}}$ and $\tilde{\alpha}_{i,j}$. However, it can be easily checked that the permutations $w''\cdot\delta_{\mathbf{hl}}$ are in fact in S; hence, they define Schubert varieties on our original $F(n_1,\dots,n_k,V)$. (If $m = i$, then $\tilde{\alpha}_{i,j}$ is the identity permutation.) Also note that for the terms appearing in the last sum, we have

$$\begin{aligned}\ell\big(w''\cdot\delta_{\mathbf{hl}}\big) &= \ell(w) + i - \sum_{c=1}^{m}\big(n_{l_c} - n_{h_c-1}\big) - \sum_{c=1}^{m}\big(n_{l_c+1} - n_{h_c}\big)\\ &= \ell(w) + \ell\big(\alpha_{i,j}\big) - \deg(q_{\mathbf{hl}}).\end{aligned}$$

(iv) Although the formula seems rather complicated, it is in fact fairly easy to compute, using the following four steps:

(1) decide which monomials $q_{\mathbf{hl}}$ may appear (these are determined by i and j, due to the conditions $m \le i$ and $h_m \le j \le l_m$);

(2) discard all $\mathbf{h},\mathbf{l}$ for which the condition $(*)$ in Remark 3.2(ii) above is not satisfied;

(3) for each of the remaining collections $\mathbf{h},\mathbf{l}$, perform the classical Pieri multiplication of $[\Omega_{w\cdot\gamma_{\mathbf{hl}}}]$ and $[\Omega_{\tilde{\alpha}_{i,j}}]$;

(4) finally, multiply (to the right) the permutations w'' obtained in the previous step by $\delta_{\mathbf{hl}}$ and discard all the ones for which the length does not drop by the number of factors in $\delta_{\mathbf{hl}}$.

For example, if $F = F(1,3,4,\mathbb{C}^5)$, $w = (53421)$ (the corresponding Schubert class is the class of a point), and $\alpha_{2,2} = (13425)$ is the cycle giving the second Chern class of the tautological quotient bundle of rank 3 on F, one computes

$$\begin{aligned}\big[\Omega_{(13425)}\big] * \big[\Omega_{(53421)}\big] &= q_2\big[\Omega_{(52431)}\big] + q_1q_2\big[\Omega_{(23541)}\big] + q_2q_3\big[\Omega_{(51423)}\big]\\ &\quad + q_1q_2q_3\big[\Omega_{(13524)}\big] + q_1q_2^2q_3\big[\Omega_{(12345)}\big].\end{aligned}$$

(v) In the case when F is the *complete* flag variety, a quantum Pieri formula is stated in the recent preprint [KiMa] of Kirillov and Maeno, and an algebraic proof is suggested. Their formulation is quite different, and we have not checked to see if it agrees with what Theorem 3.1 says in that case.

Another quantum Pieri formula, due to Lascoux and Veigneau (only for complete flags and only with an algebraic proof suggested), is announced in [Ve]. The formulation is different, but it can be seen to agree with Theorem 3.1.

After this paper was completed, Postnikov [Po] gave an algebro-combinatorial proof of quantum Pieri for complete flags, in yet another different but equivalent formulation.

We prove Theorem 3.1 in Section 6. For the moment, let us see what it says in some special cases, namely, *Grassmannians*, *complete flag varieties*, and Lemmas 3.5 and 3.6.

• *Grassmannians.* Let $k=1$ and $n_1=r$, that is, $F=G(r,n)$, the Grassmannian of r-dimensional quotients of V. Let w be a Grassmannian permutation of descent r and shape $\lambda=(\lambda_1,\dots,\lambda_r)$, with $n-r\geq\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_r\geq 0$. The partition λ is defined by $\lambda_{r-j+1}=w(j)-j$. Denote $\Omega_\lambda:=\Omega_w$. In particular, the subvariety $\Omega_{\alpha_{i,1}}$ is $\Omega_{(1^i,0^{r-i})}$ with the new notation. Finally, for a partition λ as above, let $\widetilde{\lambda}=(\widetilde{\lambda}_1,\dots,\widetilde{\lambda}_{n-r})$ be the conjugate partition given by $\widetilde{\lambda}_j=\#\{s\mid\lambda_s\geq j\}$. In this case, Theorem 3.1 translates into the following formula, due to Bertram [Be].

COROLLARY 3.3 (Quantum Pieri for Grassmannians). *One has*

$$\left[\Omega_{(1^i,0^{r-i})}\right]*\left[\Omega_\lambda\right]=\text{ (classical term) }+q\left(\sum_\mu\left[\Omega_\mu\right]\right)$$

in $QH^*(G(r,n))$, where μ ranges over partitions with at most r parts, satisfying $|\mu|=|\lambda|+i-n$ and $\widetilde{\lambda}_1-1\geq\widetilde{\mu}_1\geq\widetilde{\lambda}_2-1\geq\cdots\geq\widetilde{\mu}_{n-r-1}\geq\widetilde{\lambda}_{n-r}-1\geq\widetilde{\mu}_{n-r}=0$.

Proof. Indeed, let w be the permutation corresponding to λ. Since $k=1$, the only possible q-monomial is q itself. We have $h=l=1$, $\gamma_{h,l}=s_r\times\cdots\times s_{n-1}$, and $\delta_{h,l}=s_{r-1}\times\cdots\times s_1$. Multiplication by $\gamma_{h,l}$ transforms w into a permutation of descent $r-1$ and shape ν, with $\widetilde{\nu}_s=\widetilde{\lambda}_s-1$, for $1\leq s\leq n-r$. The classical Pieri multiplication of $[\Omega_{(1^{i-1},0^{r-i})}]$ and $[\Omega_\nu]$ gives a sum of Schubert classes indexed by partitions ρ, with

$$\widetilde{\rho}_1\geq\widetilde{\nu}_1\geq\widetilde{\rho}_2\geq\widetilde{\nu}_2\geq\cdots\geq\widetilde{\rho}_{n-r}\geq\widetilde{\nu}_{n-r}.$$

Finally, multiplication by $\delta_{h,l}$ of the permutations w'', corresponding to each ρ as above, produces permutations of descent r and shape μ, with $\widetilde{\mu}_1=\widetilde{\rho}_2$, $\widetilde{\mu}_2=\widetilde{\rho}_3,\dots,\widetilde{\mu}_{n-r-1}=\widetilde{\rho}_{n-r}$. □

• *Complete flag varieties.* Let $k=n-1$; hence, $n_j=j$ for all j, that is, $F=F(1,2,\dots,n-1,V)$. If $i=1$, then $\alpha_{1,j}=s_j$ and (3.1) specializes to the quantum Monk formula of Fomin, Gelfand, and Postnikov (see also [CF2] and [Pe]).

COROLLARY 3.4 (Quantum Monk formula). *One has in $QH^*(F)$*

$$\left[\Omega_{s_j}\right]*[\Omega_w]= \textit{(classical term)} +\sum_{t_{hl}} q_h\cdots q_{l-1}\left[\Omega_{w\cdot t_{hl}}\right],$$

where the sum is over all transpositions of integers h,l, with $1\le h\le j<l\le n$, such that $\ell(w\cdot t_{hl})=\ell(w)-2(l-h)+1$.

Finally, we now look closer at a special case, which is needed later. Recall the identity (1.4), which holds in the classical cohomology ring of our partial flag variety:

$$\left[\Omega_{\alpha_{i,j}}\right]=\sum_{r=0}^{n_j-n_{j-1}}\left(\sum_{p=0}^{r}(-1)^p\left[\Omega_{\beta_{p,j-1}}\right]\cdot\left[\Omega_{\alpha_{r-p,j}}\right]\right)\cdot\left[\Omega_{\alpha_{i-r,j-1}}\right].$$

We want to compute the right-hand side when the classical product is replaced by the quantum product. Of course, the answer is obtained by applying Theorem 3.1 twice, but this would seem to give, besides the classical term $[\Omega_{\alpha_{i,j}}]$, numerous "quantum correction" terms. In fact, a more careful analysis shows that there is either no correction term or only one such term that we explicitly identify. It is better to break the computation into two pieces.

LEMMA 3.5. (i) *In the classical cohomology ring $H^*(F)$, we have, for $0\le p\le r\le n_j-n_{j-1}$,*

$$\left[\Omega_{\beta_{p,j-1}}\right]\cdot\left[\Omega_{\alpha_{r-p,j}}\right]=\left[\Omega_{\beta_{p,j-1}\cdot\alpha_{r-p,j}}\right].$$

(ii) *In the quantum cohomology ring $QH^*(F)$, we have*

$$\left[\Omega_{\beta_{p,j-1}}\right]*\left[\Omega_{\alpha_{r-p,j}}\right]=\left[\Omega_{\beta_{p,j-1}\cdot\alpha_{r-p,j}}\right]$$

as well; that is, there are no quantum correction terms.

Proof. (i) This is a straightforward computation, for example, using Theorem 1.5 (Sottile's theorem).

(ii) Pick $1\le h_1<\cdots<h_m\le j\le l_m<\cdots<l_1\le k$. Since $n_j\ge r+n_{j-1}\ge p+n_{j-1}$, we also have $n_{l_1+1}\ge n_{j+1}\ge n_j+1\ge n_{j-1}+p+1$. Therefore, $\beta_{p,j-1}(n_{l_1+1})>\beta_{p,j-1}(m)$, for all $m<n_{l_1+1}$, by the definition of $\beta_{p,j-1}$. In particular,

$$\beta_{p,j-1}\left(n_{l_1+1}\right)>\beta_{p,j-1}\left(n_{h_1}\right). \tag{3.2}$$

To get a quantum contribution for the chosen h_i and l_i, we should have, necessarily, by Remark 3.2(ii),

$$\beta_{p,j-1}\left(n_{h_1}\right)>\max\left\{\beta_{p,j-1}\left(n_{h_1}+1\right),\ldots,\beta_{p,j-1}\left(n_{l_1+1}\right)\right\}.$$

This contradicts (3.2). $\square$

LEMMA 3.6. *The product* $[\Omega_{\beta_{p,j-1}\cdot\alpha_{r-p,j}}]*[\Omega_{\alpha_{i-r,j-1}}]$ *has no quantum correction terms, unless* $r=p=n_j-n_{j-1}$ *and* $i\geq n_j-n_{j-2}$, *in which case there is exactly one such term, namely,* $q_{j-1}[\Omega_{\alpha_{i-(n_j-n_{j-2}),j-2}}]$.

Proof. This time we need to pick $1\leq h_1<\cdots<h_m\leq j-1\leq l_m<\cdots<l_1\leq k$. If any of the h_i or l_i are different from $j-1$, the condition

$$w\big(n_{h_i}\big)>\max\big\{w\big(n_{h_i}+1\big),\dots,w\big(n_{l_i+1}\big)\big\}$$

of Remark 3.2(ii) is easily seen to be contradicted for $w:=\beta_{p,j-1}\cdot\alpha_{r-p,j}$. Hence, $m=1$, $h_1=l_1=j-1$, and we need

$$w\big(n_{j-1}\big)>\max\big\{w\big(n_{j-1}+1\big),\dots,w\big(n_j\big)\big\}.$$

This happens if and only if $r=p=n_j-n_{j-1}$; that is, the only case that may give quantum contributions is the product

$$\Big[\Omega_{\beta_{n_j-n_{j-1},j-1}}\Big]*\Big[\Omega_{\alpha_{i-(n_j-n_{j-1}),j-1}}\Big],$$

for $h=l=j-1$. In this case $\beta_{n_j-n_{j-1},j-1}\cdot\gamma_{h,l}=\mathrm{id}$ (the identity permutation), and the quantum Pieri formula (3.1) specializes to give the lemma. $\square$

From (1.3), (1.4), and the two previous lemmas, we immediately get the following corollary.

COROLLARY 3.7. *The following identities hold in* $QH^*(F)$:

$$\sigma_r^j=\sum_{p=0}^{r}(-1)^p\big[\Omega_{\beta_{p,j-1}}\big]*\big[\Omega_{\alpha_{r-p,j}}\big], \tag{3.3}$$

$$\sum_{r=0}^{n_j-n_{j-1}}\left(\sum_{p=0}^{r}(-1)^p\big[\Omega_{\beta_{p,j-1}}\big]*\big[\Omega_{\alpha_{r-p,j}}\big]\right)*\big[\Omega_{\alpha_{i-r,j-1}}\big] \tag{3.4}$$
$$=\big[\Omega_{\alpha_{i,j}}\big]+(-1)^{n_j-n_{j-1}}q_{j-1}\big[\Omega_{\alpha_{i-(n_j-n_{j-2}),j-2}}\big].$$

3.2. The quantum Giambelli formula

Definition 3.8. For $0\leq j\leq k+1$ and $1\leq i\leq n_j$, let $G_i^j\in\mathbb{Z}[\sigma,q]$ be the polynomials defined in one of the following equivalent ways.

(i) Set $G_0^j:=1$ and $G_1^j:=g_1^j$, for all j. Then G_i^j, for $i\geq 2$ and all j, is defined recursively by

$$G_i^j:=(-1)^{n_j-n_{j-1}+1}q_{j-1}G^{j-2}_{i-(n_j-n_{j-2})}+\sum_{r=0}^{n_j-n_{j-1}}\sigma_r^jG_{i-r}^{j-1}. \tag{3.5}$$

(ii) For each $1\leq j\leq k+1$, construct a graph as follows:

- choose j vertices and label them $\mathbf{v}_1, \ldots, \mathbf{v}_j$;
- for every $1 \leq l \leq j-1$, join the vertices $\mathbf{v}_l$ and $\mathbf{v}_{l+1}$ by an edge and give it the label $(-1)^{n_{l+1}-n_l+1} q_l$;
- for every $1 \leq l \leq j$, attach $n_l - n_{l-1}$ tails to the vertex $\mathbf{v}_l$, with labels $\sigma_1^l, \ldots, \sigma_{n_l - n_{l-1}}^l$, respectively.

Now define G_i^j to be the sum of all monomials obtained by choosing edges in this graph and forming the product of their labels, such that the total degree of the monomial is i, where $\deg(q_l) = n_{l+1} - n_{l-1}$ and $\deg(\sigma_m^l) = m$, for every l, m, and no two of the chosen edges share a common vertex. This description was shown to me by W. Fulton.

(iii) For each j, the polynomials G_i^j, for $1 \leq i \leq n_j$, are the coefficients of the characteristic polynomial $\det(A_{n_j}^q + \lambda I)$, where $A_{n_j}^q$ is the upper left $n_j \times n_j$ submatrix of the Astashkevich-Sadov-Kim matrix A_n^q (see [AS], [Kim1], and [Kim2]).

It is immediate from any of these descriptions that $G_i^j(\sigma, 0) = g_i^j(\sigma)$. We are now ready to formulate a special case of the quantum Giambelli formula.

THEOREM 3.9. (i) $[\Omega_{\alpha_{i,j}}] = G_i^j(\sigma, q)$ *in* $QH^*(F)$, *for all* $1 \leq i \leq n_j$ *and* $0 \leq j \leq k$.
(ii) $G_i^{k+1}(\sigma, q) = 0$ *in* $QH^*(F)$, *for all* $1 \leq i \leq n$.

Proof. The theorem follows by induction on j, using the identities (3.3) and (3.4) in Corollary 3.7, and the recursion (3.5) satisfied by the G_i^j's. □

COROLLARY 3.10 [AS], [Kim1], and [Kim2]. *There is a canonical isomorphism*

$$QH^*(F) \cong \mathbb{Z}[\sigma, q]/I_q,$$

where I_q *is the ideal* $(G_1^{k+1}, \ldots, G_n^{k+1})$.

Proof. This follows from Theorem 3.9(ii) and [ST, Theorem 2.2]. □

Remark 3.11. Theorem 3.9(ii) and Corollary 3.10 were formulated independently by Astashkevich and Sadov [AS] and Kim [Kim1], with the proof completed in [Kim2]. As far as I know, Theorem 3.9(i) is new here. For the case of complete flags, Theorem 3.9 and Corollary 3.10 were first proved in [CF1].

We now construct the polynomials that give the general quantum Giambelli formula, using an idea of Fomin, Gelfand, and Postnikov [FoGP]. For a partition $\Lambda_j := (\lambda_{j,1}, \ldots, \lambda_{j,n_{j+1}-n_j})$ with (at most) $n_{j+1} - n_j$ parts and such that each part $\lambda_{j,m}$ is at most n_j, set

$$g_{\Lambda_j}^{(j)} := g_{\lambda_{j,1}}^j g_{\lambda_{j,2}}^j \cdots g_{\lambda_{j,n_{j+1}-n_j}}^j.$$

Define *elementary monomials* $g_\Lambda := g_{\Lambda_1 \Lambda_2 \cdots \Lambda_k} \in \mathbb{Z}[\sigma]$ by

$$g_\Lambda := g_{\Lambda_1}^{(1)} g_{\Lambda_2}^{(2)} \cdots g_{\Lambda_k}^{(k)}. \tag{3.6}$$

The number of such monomials is

$$\sharp\{g_\Lambda\} = \prod_{j=0}^{k} \binom{n_{j+1}}{n_j},$$

which coincides with the rank of $H^*(F)$. It follows by realizing F as a succession of Grassmann bundles that the monomials $\{g_\Lambda\}$ generate $H^*(F)$ over $\mathbb{Z}$. Summarizing, we have the following proposition.

PROPOSITION 3.12. *The monomials $\{g_\Lambda\}$ form a $\mathbb{Z}$-basis in $H^*(F)$.*

Since the Giambelli polynomials $\{P_w(\sigma)\}_{w\in S}$ also form a basis in $H^*(F)$, we can uniquely write

$$P_w = \sum_\Lambda a_\Lambda g_\Lambda, \tag{3.7}$$

with a_Λ integers (depending, of course, on w).

From the formula (1.1), the elementary monomial g_Λ represents a cohomology class $A_\Lambda \in H^*(F)$, which is obtained as the product of k cohomology classes $B_{\Lambda_1}, \dots,$
B_{Λ_k}, and each B_{Λ_j} in turn is the product of at most $n_{j+1} - n_j$ factors of type $[\Omega_{\alpha_{i,j}}]$, for various i's. From Theorem 1.4, we get

$$[\Omega_w] = \sum_\Lambda a_\Lambda A_\Lambda. \tag{3.8}$$

The following definition is a straightforward extension of the one given for the case of complete flags in [FoGP].

Definition 3.13. The *quantum elementary monomial* G_Λ is the polynomial in $\mathbb{Z}[\sigma, q]$, obtained by replacing each factor g_i^j in g_Λ by the corresponding G_i^j, from Definition 3.8.

The usefulness of the quantum elementary monomials is highlighted by the next result, which is given a geometric proof in Section 6.

THEOREM 3.14. *Let $w_1, \dots, w_N \in S$ be a collection of permutations satisfying the conditions*

(1) *each w_m, for $1 \le m \le N$, is a cycle $\alpha_{i,j}$, for some i and j;*

(2) *for each j, the number of cycles $\alpha_{i,j}$ among the w_m's is at most $n_{j+1} - n_j$.*

Then

$$[\Omega_{w_1}] * \cdots * [\Omega_{w_N}] = [\Omega_{w_1}] \times \cdots \times [\Omega_{w_N}];$$

that is, the quantum product $[\Omega_{w_1}] * \cdots * [\Omega_{w_N}]$ has no q-terms.

Definition 3.15. The *quantum Giambelli polynomial* $P_w^q \in \mathbb{Z}[\sigma, q]$ is defined by

$$P_w^q := \sum_{\Lambda} a_\Lambda G_\Lambda, \tag{3.9}$$

with a_Λ the integers from (3.7).

THEOREM 3.16 (Quantum Giambelli formula). *$[\Omega_w] = P_w^q(\sigma, q)$ in $QH^*(F)$, for all $w \in S$.*

Proof. By Theorem 3.9(i), the quantum elementary monomial G_Λ represents the *quantum* product obtained by replacing each G_i^j in G_Λ by the corresponding $[\Omega_{\alpha_{i,j}}]$. It follows from Theorem 3.14 that this quantum product is equal to the cohomology class A_Λ in $QH^*(F)$. Hence, the quantum Giambelli polynomial $P_w^q(\sigma, q)$ represents the class $\sum_\Lambda a_\Lambda A_\Lambda$ in $QH^*(F)$. By (3.8), this class coincides with $[\Omega_w]$. □

COROLLARY 3.17 (Bertram [Be]). *For the Grassmannian $G(r, n)$, the classical and quantum Giambelli formulas are the same.*

Proof. By Definition 3.8, we have $G_i^1 = g_i^1$, for all $1 \le i \le r$; hence, the quantum Giambelli polynomials coincide with the classical ones. □

Example 3.18. Table 1 shows the quantum Giambelli polynomials for the partial flag variety $F(1, 3, \mathbb{C}^4)$. $F(1, 3, \mathbb{C}^4)$ has dimension 5, and the Schubert varieties are indexed by the twelve permutations $w \in S_4$ satisfying $w(2) < w(3)$. The σ-variables are

$$\sigma_1^1 := x_1, \qquad \sigma_1^2 := x_2 + x_3, \qquad \sigma_2^2 := x_2 x_3, \qquad \sigma_1^3 := x_4.$$

We have two quantum parameters q_1 and q_2, both of degree 3; however, only q_1 appears in the polynomials P_w^q. (This is a general fact: It follows from Definitions 3.8 and 3.15 that q_k does not appear in the quantum Giambelli polynomials for $F(n_1, \ldots, n_k, V)$.) The quantum elementary monomials are the products formed by taking one factor from

$$\left\{1, G_1^1, \left(G_1^1\right)^2\right\}$$

and one factor from

$$\left\{1, G_1^2, G_2^2, G_3^2\right\}.$$

3.3. Further properties and results. The following is easily obtained from Definitions 3.8, 3.13, and 3.15.

PROPOSITION 3.19. (i) *$P_w^q(\sigma, q)$ is a weighted homogeneous polynomial of weighted degree $\ell(w)$, where σ_i^j has degree i, and q_j has degree $n_{j+1} - n_{j-1}$, for all $1 \le j \le k$.*

(ii) *$P_w^q(\sigma, 0) = P_w(\sigma)$.*

(iii) *$\{G_\Lambda\}$ and $\{P_w^q\}$ are linear bases for $QH^*(F)$ (cf. [FoGP, Propositions 3.6 and 3.7]).*

TABLE 1

Quantum Giambelli polynomials for $F\big(1,3,\mathbb{C}^4\big)$

w	P_w^q in the basis $\{G_\Lambda\}$	$P_w^q(\sigma,q)$
(1234)	1	1
(2134)	G_1^1	σ_1^1
(1243)	G_1^2	$\sigma_1^1+\sigma_1^2$
(1342)	G_2^2	$\sigma_1^1\sigma_1^2+\sigma_2^2$
(2143)	$G_1^1G_1^2$	$\big(\sigma_1^1\big)^2+\sigma_1^1\sigma_1^2$
(3124)	$\big(G_1^1\big)^2$	$\big(\sigma_1^1\big)^2$
(3142)	$G_1^1G_2^2-G_3^2$	$\big(\sigma_1^1\big)^2\sigma_1^2+q_1$
(2341)	G_3^2	$\sigma_1^1\sigma_2^2-q_1$
(4123)	$\big(G_1^1\big)^2G_1^2-G_1^1G_2^2+G_3^2$	$\big(\sigma_1^1\big)^3-q_1$
(3241)	$G_1^1G_3^2$	$\big(\sigma_1^1\big)^2\sigma_2^2-\sigma_1^1q_1$
(4132)	$\big(G_1^1\big)^2G_2^2-G_1^1G_3^2$	$\big(\sigma_1^1\big)^3\sigma_1^2+\sigma_1^1q_1$
(4231)	$\big(G_1^1\big)^2G_3^2$	$\big(\sigma_1^1\big)^3\sigma_2^2-\big(\sigma_1^1\big)^2q_1$

• *Operator definition of quantum Giambelli polynomials.* The quantum Giambelli polynomials can also be defined via divided difference operators similar to the ones used in the classical case. For the quantum Schubert polynomials, this was done in [KiMa] (see also [CFF] and [F3]). The idea is to define double versions of these polynomials, using an additional set of variables $y_1,\dots,y_n$, such that when the y variables are set equal to zero, one recovers the quantum Giambelli polynomials.

Let $w^\circ\in S$ be the longest element, given by $w(i)=n-n_j+i-n_{j-1}$, for all $n_{j-1}+1\le i\le n_j$, $1\le j\le k+1$. Its length is $\ell(w^\circ)=\sum_{j=1}^k(n-n_j)(n_j-n_{j-1})=\dim F$ and $[\Omega_{w^\circ}]$ is the class of a point in $H_0(F)$. By the classical Giambelli formula,

$$\big[\Omega_{w^\circ}\big]=P_{w^\circ}(\sigma)=\big(\sigma_{n_1}^1\big)^{n-n_1}\big(\sigma_{n_2-n_1}^2\big)^{n-n_2}\cdots\big(\sigma_{n_k-n_{k-1}}^k\big)^{n-n_k},$$

while expressing $[\Omega_{w^\circ}]$ in the basis $\{g_\Lambda\}$ yields

$$\big[\Omega_{w^\circ}\big] = \underbrace{g^1_{n_1} g^1_{n_1} \cdots g^1_{n_1}}_{n_2-n_1 \text{ factors}} \underbrace{g^2_{n_2} g^2_{n_2} \cdots g^2_{n_2}}_{n_3-n_2 \text{ factors}} \cdots \underbrace{g^k_{n_k} g^k_{n_k} \cdots g^k_{n_k}}_{n_{k+1}-n_k \text{ factors}} = g_{\Lambda^\circ},$$

where

$$\Lambda^\circ = \big(\Lambda_1, \dots, \Lambda_k\big), \qquad \Lambda_j = (\underbrace{n_j, \dots, n_j}_{n_{j+1}-n_j \text{ terms}}).$$

For all integers m, and all $j \in \{1, 2, \dots, k\}$, set

$$f_m(j) := \sum_{r=0}^{m} (-1)^r G^j_{m-r} h_r\big(y_{n-n_{j+1}+1}, \dots, y_{n-n_j}\big),$$

where $G^j_i = G^j_i(\sigma, q)$ are the polynomials in Definition 3.8 for $i \le n_j$, $G^j_i = 0$ for $i > n_j$, and $h_r(y_{n-n_{j+1}+1}, \dots, y_{n-n_j})$ is the rth complete symmetric function in the indicated variables. Now set

$$D(j) := \det\big(f_{n_j+b-a}(j)\big)_{1 \le a,b \le n_{j+1}-n_j},$$

and define the *quantum double Giambelli polynomial of* w° to be

$$P_{w^\circ}(\sigma, q, y) := \prod_{j=1}^{k} D(j).$$

For $1 \le l \le n$, ∂^y_l denotes the divided difference operator acting on polynomials $P(\sigma, q, y)$ by

$$\partial^y_l P(\sigma, q, y) = \frac{P(\sigma, q, y) - s^y_l P(\sigma, q, y)}{y_{l+1} - y_l},$$

with the operator s^y_l interchanging y_l and y_{l+1}.

For every permutation $w \in S$, one can write $w = s_{l_t} \times \cdots \times s_{l_1} \cdot w^\circ$, with $t = \ell(w^\circ) - \ell(w)$. We define the *quantum double Giambelli polynomial* $P_w(\sigma, q, y)$ by

$$P_w(\sigma, q, y) := (-1)^t \partial^y_{l_t} \circ \cdots \circ \partial^y_{l_1}\big(P_{w^\circ}(\sigma, q, y)\big).$$

A proof of the next proposition can be found in [F3, Section 4] (cf. [KiMa], [CFF]).

PROPOSITION 3.20. *The polynomial* $P_w(\sigma, q, 0)$ *coincides with the quantum Giambelli polynomial* P^q_w *in Definition 3.15.*

• *Orthogonality.* For a polynomial $R \in \mathbb{Z}[\sigma]$, consider the expansion of its coset $R(\operatorname{mod} I) \in \mathbb{Z}[\sigma]/I$ in the basis $\{P_w\}$ and define

$$\langle R \rangle := \text{ coefficient of } P_{w^\circ}.$$

Alternately, expand $R(\operatorname{mod} I)$ in the basis $\{g_\Lambda\}$ and take the coefficient of g_{Λ°. By the classical Giambelli formula (Theorem 1.3), we can reformulate Theorem 1.2 as the following.

PROPOSITION 3.21. *The polynomials $\{P_w\}$ satisfy the orthogonality property*

$$\langle P_w P_v \rangle = \begin{cases} 1, & \textit{if } v = \check{w}, \\ 0, & \textit{otherwise}, \end{cases}$$

where $\check{w}$ is the permutation giving the Schubert variety dual to $[\Omega_w]$.

Similarly, for $R(\sigma, q) \in \mathbb{Z}[\sigma, q]$, consider the expansion of $R(\operatorname{mod} I_q)$ in the basis $\{P_w^q\}$ (or $\{G_\Lambda\}$, respectively) and define

$$\langle\langle R \rangle\rangle := \text{ coefficient of } P^q_{w^\circ} (\text{ or } G_{\Lambda^\circ}, \text{ respectively}). \tag{3.10}$$

As a consequence of Theorem 3.16, we get the following.

THEOREM 3.22 (Orthogonality of the quantum Giambelli polynomials). *We have*

$$\left\langle\left\langle P_w^q P_v^q \right\rangle\right\rangle = \begin{cases} 1, & \textit{if } v = \check{w}, \\ 0, & \textit{otherwise}. \end{cases}$$

Remark 3.23. For the special case of complete flags, the above theorem is due to [FoGP] and was proved using combinatorial techniques, *without* the use of the quantum Giambelli formula. In fact, the proof of the quantum Giambelli formula in their paper follows from Theorem 3.9(i) and orthogonality.

• *Quantum Pieri for* $[\Omega_{\beta_{i,j}}]$. For every flag variety $F = F(n_1, \ldots, n_k, V)$, there is a canonical isomorphism $F \xrightarrow{\cong} F^*$, where $F^* = F(n - n_k, \ldots, n - n_1, V^*)$ denotes the *dual* flag variety. This isomorphism sends the flag of quotients

$$V \twoheadrightarrow U_k \twoheadrightarrow \cdots \twoheadrightarrow U_1$$

to the flag

$$V^* \twoheadrightarrow W_k \twoheadrightarrow \cdots \twoheadrightarrow W_1,$$

with W_j the dual of $\ker(V \twoheadrightarrow U_{k-j+1})$. There are induced isomorphisms

$$\delta : H^*(F) \xrightarrow{\cong} H^*\big(F^*\big), \qquad \delta_q : QH^*(F) \xrightarrow{\cong} QH^*\big(F^*\big),$$

on the classical and quantum cohomology rings. The map δ_q acts on the Schubert basis by $\delta_q([\Omega_w]) = [\Omega_{w_\circ w w_\circ}]$ and on the q variables by $\delta_q(q_j) = q_{k-j}$. (Recall that $w_\circ$ is the longest permutation in the symmetric group S_n, given by $w_\circ(i) = n-i+1$.) In particular, one gets $\delta_q([\Omega_{\beta_{i,j}}]) = [\Omega_{\alpha_{i,k-j+1}}]$.

Hence, in complete analogy with the classical case, the Pieri formula for quantum multiplication with the classes $[\Omega_{\alpha_{i,j}}]$ on F^* (Theorem 3.1) gives a formula for quantum multiplication with the classes $[\Omega_{\beta_{i,j}}]$ on F. The proofs of the above statements, as well as writing down the actual formula, are straightforward and left to the reader.

We have completed the description of $QH^*(F)$, modulo the proofs of Theorems 3.1 and 3.14. The last three sections of the paper are devoted to these proofs, which are based on the geometry of compactifications of spaces of maps $\mathbb{P}^1 \to F$ given by hyperquot schemes. Most of the arguments in [CF2], where the case of complete flags is treated, require little or no changes. Therefore, we refer to the corresponding results in [CF2] when appropriate and give details only as needed.

4. GW invariants via hyperquot schemes. We recall in this section the construction of 3-point, genus-zero GW invariants by means of hyperquot schemes. Details can be found in [Be] and [CF2].

4.1. Hom *and hyperquot schemes.* For fixed $\overline{d} = (d_1, d_2, \dots, d_k)$, let $H_{\overline{d}} := \mathrm{Hom}_{\overline{d}}(\mathbb{P}^1, F)$ be the moduli space of holomorphic maps $\mu : \mathbb{P}^1 \to F$ of multidegree $\overline{d}$; that is, such that $\mu_*[\mathbb{P}^1] = \sum_{j=1}^k d_j [\Omega_{\check{s}_{n_j}}]$. $H_{\overline{d}}$ is a quasi-projective scheme, as it can be embedded as an open subscheme of the Hilbert scheme of $\mathbb{P}^1 \times F$. Since F is a homogeneous space, $H^1(\mathbb{P}^1, \mu^* T_F) = 0$ for every map μ, and standard deformation theory shows that $H_{\overline{d}}$ is smooth and of dimension

$$\begin{aligned} h^0\bigl(\mathbb{P}^1, \mu^* T_F\bigr) &= \dim F - \mu_*\bigl[\mathbb{P}^1\bigr] \cdot \bigl(K_F\bigr) \\ &= \sum_{j=1}^k \bigl(n - n_j\bigr)\bigl(n_j - n_{j-1}\bigr) + \sum_{j=1}^k d_j \bigl(n_{j+1} - n_{j-1}\bigr). \end{aligned}$$

To give a map of multidegree $\overline{d}$ is equivalent to specifying a sequence of quotient bundles

$$V_{\mathbb{P}^1} \twoheadrightarrow M_k \twoheadrightarrow \cdots \twoheadrightarrow M_1,$$

with rank $M_j = n_j$ and $\deg(M_j) = d_j$, or, by dualizing, a sequence of subbundles

$$S_1 \subset \cdots \subset S_k \subset V^*_{\mathbb{P}^1},$$

with rank $S_j = n_j$ and $\deg(S_j) = -d_j$. Let $T_j := V^*_{\mathbb{P}^1}/S_{k-j+1}$. The Hilbert polynomial of T_j is $P_j(m) = (m+1)(n - n_{k-j+1}) + d_{k-j+1}$.

Let $\mathcal{HQ}_{\overline{d}} := \mathcal{HQ}_{P_1,\dots,P_k}(\mathbb{P}^1, V^*_{\mathbb{P}^1})$ be the hyperquot scheme parametrizing flagged sequences of quotient *sheaves* of $V^*_{\mathbb{P}^1}$, with Hilbert polynomials given by $P_1, \dots, P_k$.

THEOREM 4.1 [Lau], [CF2], and [Kim3]. (i) *$\mathcal{HQ}_{\overline{d}}$ is a smooth, irreducible, projective variety, of dimension*

$$\sum_{j=1}^{k}(n-n_j)(n_j-n_{j-1})+\sum_{j=1}^{k}d_j(n_{j+1}-n_{j-1}),$$

containing $H_{\overline{d}}$ as an open dense subscheme.

(ii) *$\mathcal{HQ}_{\overline{d}}$ is a fine moduli space; that is, there exists a universal sequence*

$$(\dagger)\qquad V^*_{\mathbb{P}^1\times\mathcal{HQ}_{\overline{d}}}\twoheadrightarrow \mathcal{T}_k^{\overline{d}}\twoheadrightarrow\cdots\twoheadrightarrow\mathcal{T}_2^{\overline{d}}\twoheadrightarrow\mathcal{T}_1^{\overline{d}}\twoheadrightarrow 0$$

on $\mathbb{P}^1\times\mathcal{HQ}_{\overline{d}}$ such that each $\mathcal{T}_j^{\overline{d}}$ is flat over $\mathcal{HQ}_{\overline{d}}$, with relative Hilbert polynomial $P_j(m)$. The sequence has the property that for every scheme X over $\mathbb{C}$, together with a sequence of quotients

$$(\dagger\dagger)\qquad V^*_{\mathbb{P}^1\times X}\twoheadrightarrow Q_k\twoheadrightarrow\cdots\twoheadrightarrow Q_1$$

such that each Q_j is flat over X, with relative Hilbert polynomial P_j, there exists a unique morphism $\Phi_X : X\to\mathcal{HQ}_{\overline{d}}$ such that sequence (††) is the pullback of (†) via (id,Φ_X).

(iii) *Let $\mathcal{S}_j^{\overline{d}} := \ker(V^*_{\mathbb{P}^1\times\mathcal{HQ}_{\overline{d}}}\to\mathcal{T}^{\overline{d}}_{k-j+1})$. Then $\mathcal{S}_j^{\overline{d}}$ is a* vector bundle *of rank n_j and relative degree $-d_j$ on $\mathbb{P}^1\times\mathcal{HQ}_{\overline{d}}$.*

4.2. Generalized Schubert varieties on $H_{\overline{d}}$ and $\mathcal{HQ}_{\overline{d}}$. The moduli space of maps comes with a universal evaluation morphism

$$ev:\mathbb{P}^1\times H_{\overline{d}}\to F,$$

given by $ev(t,[\mu])=\mu(t)$, which can be used to pull back Schubert varieties to $H_{\overline{d}}$. More precisely, for $t\in\mathbb{P}^1$ and $w\in S$, define a subscheme of $H_{\overline{d}}$ by

$$\Omega_w(t)=ev^{-1}\left(\Omega_w\right)\bigcap\left(\{t\}\times H_{\overline{d}}\right).$$

As a set, $\Omega_w(t)$ consists of all maps of multidegree $\overline{d}$, sending $t\in\mathbb{P}^1$ to Ω_w.

Alternately, the pullback $\Omega_w(t)$ of a Schubert variety can be described as the degeneracy locus

$$\left\{\operatorname{rank}\left(V_p\otimes\mathcal{O}\to ev^*Q_q\right)\le r_w(q,p),1\le p\le n,q\in\{n_1,\dots,n_k\}\right\}\bigcap\left(\{t\}\times H_{\overline{d}}\right),$$

where the Q_j's are the tautological quotient bundles on F and where $V_1\subset\cdots\subset V_{n-1}\subset V_n=V$ is our fixed reference flag. This last description may be used to extend $\Omega_w(t)$ to $\mathcal{HQ}_{\overline{d}}$.

Definition 4.2. $\overline{\Omega}_w(t)$ is the subscheme of $\mathcal{HQ}_{\overline{d}}$, defined as the degeneracy locus

$$\left\{\operatorname{rank}\left(V_p\otimes\mathcal{O}\to\left(\mathcal{S}_q^{\overline{d}}\right)^*\right)\le r_w(q,p),1\le p\le n,q\in\left\{n_1,\dots,n_k\right\}\right\}\bigcap\left(\{t\}\times\mathcal{HQ}_{\overline{d}}\right).$$

4.3. GW invariants. To define the GW invariants, we need the following theorem.

THEOREM 4.3 (Moving lemma). (i) *Let Y be a fixed subvariety of $H_{\overline{d}}$. For any $w\in S$, a corresponding general translate of $\Omega_w\subset F$, and $t\in\mathbb{P}^1$, the intersection $Y\bigcap\Omega_w(t)$ is either empty or has pure codimension $\ell(w)$ in Y. In particular, for any $w_1,\dots,w_N\in S$, $t_1,\dots,t_N\in\mathbb{P}^1$, and general translates of $\Omega_{w_i}\subset F$, the intersection $\bigcap_{i=1}^N\Omega_{w_i}(t_i)$ is either empty or has pure codimension $\sum_{i=1}^N\ell(w_i)$ in $H_{\overline{d}}$.*

(ii) *Moreover, if $t_1,\dots,t_N$ are* distinct, *then for general translates of Ω_{w_i}, the intersection $\bigcap_{i=1}^N\overline{\Omega}_{w_i}(t_i)$ is either empty or has pure codimension $\sum_{i=1}^N\ell(w_i)$ in $\mathcal{HQ}_{\overline{d}}$ and is the Zariski closure of $\bigcap_{i=1}^N\Omega_{w_i}(t_i)$.*

Proof. (i) This follows from Kleiman's transversality theorem [Kl], since F is a homogeneous space.

(ii) The proof is given in Section 6. □

In particular, $\overline{\Omega}_w(t)$ is the closure of $\Omega_w(t)$ in $\{t\}\times\mathcal{HQ}_{\overline{d}}$, and it has pure codimension $\ell(w)$; hence, via the identification $\{t\}\times\mathcal{HQ}_{\overline{d}}\cong\mathcal{HQ}_{\overline{d}}$, it determines a class $[\overline{\Omega}_w(t)]\in H^{2\ell(w)}(\mathcal{HQ}_{\overline{d}})$. From the above theorem, we immediately get the following corollary (see, e.g., [Be]).

COROLLARY 4.4. (i) *The class $[\overline{\Omega}_w(t)]$ in the cohomology (or Chow) ring of $\mathcal{HQ}_{\overline{d}}$ is independent of $t\in\mathbb{P}^1$ and the flag $V_\bullet\subset V$.*

(ii) *Assume that $\sum_{i=1}^N\ell(w_i)=\dim(H_{\overline{d}})$. Then, as long as $t_1,\dots,t_N$ are distinct and we pick general translates of the subvarieties $\Omega_{w_i}\subset F$, the number of points in $\bigcap_{i=1}^N\Omega_{w_i}(t_i)=\bigcap_{i=1}^N\overline{\Omega}_{w_i}(t_i)$ is equal to the intersection number*

$$\int_{\mathcal{HQ}_{\overline{d}}}\left[\overline{\Omega}_{w_1}(t_1)\right]\cup\dots\cup\left[\overline{\Omega}_{w_N}(t_N)\right].$$

Definition 4.5. The Gromov-Witten invariant associated to the Schubert classes $[\Omega_{w_1}],\dots,[\Omega_{w_N}]$ is

$$\left\langle\Omega_{w_1},\dots,\Omega_{w_N}\right\rangle_{\overline{d}}:=\int_{\mathcal{HQ}_{\overline{d}}}\left[\overline{\Omega}_{w_1}(t_1)\right]\cup\dots\cup\left[\overline{\Omega}_{w_N}(t_N)\right],$$

with $t_1,\dots,t_N\in\mathbb{P}^1$.

By Corollary 4.4, for general translates of $\Omega_{w_1},\dots,\Omega_{w_N}$ and distinct $t_1,\dots,t_N$,

$$\langle\Omega_{w_1},\dots,\Omega_{w_N}\rangle_{\overline{d}}:=\begin{cases}\operatorname{card}\left(\bigcap_{i=1}^N\Omega_{w_i}(t_i)\right), & \text{if }\sum_{i=1}^N\ell(w_i)=\dim(H_{\overline{d}}),\\ 0, & \text{otherwise}.\end{cases}$$

From Lemma 2.1, we get the following corollary.

COROLLARY 4.6. *The invariant* $I_{3,\overline{d}}([\Omega_{w_1}][\Omega_{w_2}][\Omega_{w_3}])$, *defined using Kontsevich's space of stable maps* $\overline{M}_{0,3}(F,\overline{d})$, *coincides with* $\langle\Omega_{w_1},\Omega_{w_2},\Omega_{w_3}\rangle_{\overline{d}}$. *Hence,*

$$[\Omega_u]*[\Omega_v]=\sum_{\overline{d}} q_1^{d_1}\cdots q_k^{d_k}\sum_{w\in S}\langle\Omega_u,\Omega_v,\Omega_w\rangle_{\overline{d}}[\Omega_{\check{w}}].$$

When $N>3$, $\langle\Omega_{w_1},\dots,\Omega_{w_N}\rangle_{\overline{d}}$ does *not* coincide with the GW invariant $I_{n,\overline{d}}([\Omega_{w_1}]\cdots[\Omega_{w_N}])$ of [KM], as they are solutions to different enumerative geometry problems. However, $\langle\Omega_{w_1},\dots,\Omega_{w_N}\rangle_{\overline{d}}$ can also be realized as an intersection number on the Kontsevich moduli space $\overline{M}_{0,N}(F,\overline{d})$, and, from this definition, it is not hard to see that

$$\langle\Omega_{w_1},\dots,\Omega_{w_N}\rangle_{\overline{d}}=\sum_{\overline{e}+\overline{f}=\overline{d}}\sum_{v\in S}\langle\Omega_{w_1},\dots,\Omega_{w_{N-2}},\Omega_v\rangle_{\overline{e}}\langle\Omega_{\check{v}},\Omega_{w_{N-1}},\Omega_{w_N}\rangle_{\overline{f}}, \tag{4.1}$$

from which it follows that

$$[\Omega_{w_1}]*[\Omega_{w_2}]*\cdots*[\Omega_{w_N}]=\sum_{\overline{d}} q_1^{d_1}\cdots q_k^{d_k}\sum_{w\in S}\langle\Omega_{w_1},\dots,\Omega_{w_N},\Omega_w\rangle_{\overline{d}}[\Omega_{\check{w}}]. \tag{4.2}$$

We use (4.2) in Section 6 for the proof of Theorem 3.14.

Remark 4.7. (i) Sometimes (4.2) is used as the definition of the quantum multiplication in the small quantum cohomology ring, and then (4.1) is equivalent to the associativity of this multiplication.

(ii) (Compare [FoGP].) The relation (4.2), together with the quantum Giambelli formula and the orthogonality property of the quantum Giambelli polynomials, gives a direct (algebro-combinatorial) method to compute the GW invariants of F. Namely, for any $w_1,\dots,w_N\in S$, we get

$$\sum_{\overline{d}} q_1^{d_1}\cdots q_k^{d_k}\langle\Omega_{w_1},\dots,\Omega_{w_N}\rangle_{\overline{d}}=\langle\langle P^q_{w_1}\times\cdots\times P^q_{w_N}\rangle\rangle, \tag{4.3}$$

where $\langle\langle\cdots\rangle\rangle$ is defined by (3.10). The Gröbner basis techniques for more efficient computations of the invariants, as described in [FoGP, Section 12] for the case of complete flags, can be readily extended to cover the partial flag varieties as well.

We conclude this section by recording for later use a result similar to Theorem 4.3, due to Kim [Kim3, Corollary 3.2].

For every irreducible closed subvariety $Y\subset F$ and every $t\in\mathbb{P}^1$, we denote by $Y(t)$ the preimage $ev^{-1}(Y)\bigcap\{t\}\times H_{\overline{d}}$ and by $\overline{Y(t)}$ the closure of $Y(t)$ in $\{t\}\times\mathcal{HQ}_{\overline{d}}$.

PROPOSITION 4.8. *Let* $Y_1,\dots,Y_N$ *be closed, irreducible subvarieties in* F, *and* *let $t_1, \ldots, t_N$ be distinct points in $\mathbb{P}^1$. Assume that $\sum_{i=1}^N \operatorname{codim} Y_i = \dim H_{\overline{d}}$. Then for general translates of Y_i, the intersection scheme $\bigcap_{i=1}^N Y_i(t_i)$ is either empty or consists of finitely many reduced points. Moreover,*

$$\bigcap_{i=1}^N Y_i(t_i) = \bigcap_{i=1}^N \overline{Y_i(t_i)},$$

and the cardinality of this set is equal to the intersection number

$$\int_{\mathcal{HQ}_{\overline{d}}} \left[\overline{Y_1(t_1)}\right] \cup \cdots \cup \left[\overline{Y_N(t_N)}\right].$$

Note that Theorem 4.3 does not follow from Proposition 4.8, since $\overline{\Omega}_w(t)$ may be a priori larger than the closure of $\Omega_w(t)$.

5. The boundary of $\mathcal{HQ}_{\overline{d}}$. The space $H_{\overline{d}}$ is the largest subscheme of $\mathcal{HQ}_{\overline{d}}$ with the property that on $\mathbb{P}^1 \times H_{\overline{d}}$ the sheaf injections in the universal sequence

$$0 \hookrightarrow \mathcal{S}_1^{\overline{d}} \hookrightarrow \mathcal{S}_2^{\overline{d}} \hookrightarrow \cdots \hookrightarrow \mathcal{S}_k^{\overline{d}} \hookrightarrow V^*_{\mathbb{P}^1 \times \mathcal{HQ}_{\overline{d}}}$$

are vector bundle inclusions. The boundary of $\mathcal{HQ}_{\overline{d}}$, by which we mean the complement of $H_{\overline{d}}$, is therefore the locus $\mathcal{B}_{\overline{d}}$ such that on $\mathbb{P}^1 \times \mathcal{B}_{\overline{d}}$ some of these maps degenerate. In this section we study the restrictions of the generalized Schubert varieties $\overline{\Omega}_w(t)$ to $\mathcal{B}_{\overline{d}}$. We start with a description of $\mathcal{B}_{\overline{d}}$ itself, for which the following construction (taken from [CF2, Construction 2.2]) is needed.

Let $\overline{e} = (e_1, \ldots, e_k)$ be a multi-index satisfying the conditions

$$e_i \le \min(n_i, d_i), \quad \text{for } 1 \le i \le k, \tag{5.1}$$

$$e_i - e_{i-1} \le n_i - n_{i-1}, \quad \text{for } 2 \le i \le k \tag{5.2}$$

(cf. [CF2, Lemma 2.1]).

For each $1 \le i \le k$, let $\pi_i : \mathcal{G}_i \longrightarrow \mathbb{P}^1 \times \mathcal{HQ}_{\overline{d}-\overline{e}}$ be the Grassmann bundle of e_i-dimensional quotients of $\mathcal{S}_i^{\overline{d}-\overline{e}}$. Let $\mathcal{G}_{\overline{e}}$ be the fiber product of the $\mathcal{G}_i$'s over $\mathbb{P}^1 \times \mathcal{HQ}_{\overline{d}-\overline{e}}$, with projection $\pi : \mathcal{G}_{\overline{e}} \longrightarrow \mathbb{P}^1 \times \mathcal{HQ}_{\overline{d}-\overline{e}}$.

For each $1 \le i \le k$, let

$$0 \longrightarrow K_i \longrightarrow \pi_i^* \mathcal{S}_i^{\overline{d}-\overline{e}} \longrightarrow L_i \longrightarrow 0$$

be the universal sequence on $\mathcal{G}_i$. K_i and L_i are vector bundles of ranks $n_i - e_i$ and e_i, respectively. On $\mathcal{G}_{\overline{e}}$ we have the following diagram:

$$\begin{tikzcd}
0 \arrow[d] & \cdots & 0 \arrow[d] & 0 \arrow[d] & \cdots & 0 \arrow[d] & \\
K_1 \arrow[d] & \cdots & K_i \arrow[d] & K_{i+1} \arrow[d] & \cdots & K_k \arrow[d] & \\
\pi^*\mathcal{S}_1^{\bar d-\bar e} \arrow[r] \arrow[d] & \cdots \arrow[r] & \pi^*\mathcal{S}_i^{\bar d-\bar e} \arrow[r] \arrow[d] & \pi^*\mathcal{S}_{i+1}^{\bar d-\bar e} \arrow[r] \arrow[d] & \cdots \arrow[r] & \pi^*\mathcal{S}_k^{\bar d-\bar e} \arrow[r] \arrow[d] & V^*_{\mathcal{G}_{\bar e}} \\
L_1 \arrow[d] & \cdots & L_i \arrow[d] & L_{i+1} \arrow[d] & \cdots & L_k \arrow[d] & \\
0 & \cdots & 0 & 0 & \cdots & 0. &
\end{tikzcd}$$

Let $\mathcal{U}_{\bar e}$ be the locally closed subscheme of $\mathcal{G}_{\bar e}$ determined by the closed conditions

$$\operatorname{rank}\big(K_i \longrightarrow L_{i+1}\big) = 0, \qquad \text{for } i = 1, \ldots, k-1, \tag{5.3}$$

and the open conditions

$$\operatorname{rank}\big(K_i \longrightarrow V^*_{\mathcal{G}_{\bar e}}\big) = n_i - e_i, \qquad \text{for } i = 1, \ldots, k. \tag{5.4}$$

LEMMA 5.1. *$\mathcal{U}_{\bar e}$ is smooth, irreducible, and of dimension*

$$1 + \dim\big(\mathcal{HQ}_{\bar d}\big) - \sum_{i=1}^{k} e_i\big(n_{i+1} - n_i\big) - \sum_{i=1}^{k} e_i(e_i - e_{i-1}).$$

The projection $\pi : \mathcal{U}_{\bar e} \to \mathbb{P}^1 \times \mathcal{HQ}_{\bar d-\bar e}$ is smooth, and its image contains $\mathbb{P}^1 \times H_{\bar d-\bar e}$.

Proof. For a vector bundle E on a scheme X, we denote by $G_e(E)$ the Grassmann bundle of e-dimensional quotients of E, for some $0 \le e \le \operatorname{rank}(E)$.

Let $\mathcal{V}_{\bar e} \subset \mathcal{G}_{\bar e}$ be the open subscheme defined by the conditions (5.4), and put $\mathcal{V} := \pi(\mathcal{V}_{\bar e})$. Obviously, $\mathcal{V}$ is open in $\mathbb{P}^1 \times \mathcal{HQ}_{\bar d-\bar e}$ and contains $\mathbb{P}^1 \times H_{\bar d-\bar e}$. The lemma is a consequence of the observation that $\mathcal{U}_{\bar e}$ can be constructed as a sequence of k Grassmann bundles over $\mathcal{V}$ as follows:

- start with $\rho_1 : G_{e_1}(\mathcal{S}_1^{\bar d-\bar e}) \to \mathcal{V}$ with universal subbundle K_1;
- next, form $\rho_2 : G_{e_2}(\rho_1^*\mathcal{S}_2^{\bar d-\bar e}/K_1) \to G_{e_1}(\mathcal{S}_1^{\bar d-\bar e})$ with universal subbundle $\mathcal{L}_2$, and let K_2 be the natural induced extension

$$0 \to \rho_2^* K_1 \to K_2 \to \mathcal{L}_2 \to 0;$$

- continue by forming $\rho_3 : G_{e_3}(\rho_2^*\rho_1^*\mathcal{S}_3^{\bar d-\bar e}/K_2) \to G_{e_2}(\rho_1^*\mathcal{S}_2^{\bar d-\bar e}/K_1)$, with universal subbundle $\mathcal{L}_3$, and let K_3 be the natural extension

$$0 \to \rho_3^* K_2 \to K_3 \to \mathcal{L}_3 \to 0,$$

and so on.

We use this description of $\mathcal{U}_{\overline{e}}$ in the proof of quantum Pieri (see Section 6.3). $\square$

THEOREM 5.2. *There exist morphisms $\phi_{\overline{e}} : \mathcal{U}_{\overline{e}} \longrightarrow \mathcal{HQ}_{\overline{d}}$ with the following properties.*

(i) *If* $\operatorname{rank}_{(t,x)} \mathcal{T}^{\overline{d}}_{k-i+1} = n - n_i + e_i$ *for every* $1 \le i \le k$ *at a point* $(t, x) \in \mathbb{P}^1 \times \mathcal{HQ}_{\overline{d}}$, *then* $x \in \phi_{\overline{e}}(\mathcal{U}_{\overline{e}})$. *In particular,* $\mathcal{B}_{\overline{d}}$ *is covered by the union of* $\phi_{\overline{e}}(\mathcal{U}_{\overline{e}})$, *where* $\overline{e}$ *ranges over all (nonzero) multi-indices satisfying (5.1) and (5.2).*

(ii) *The restriction of* $\phi_{\overline{e}}$ *to* $\pi^{-1}(\mathbb{P}^1 \times H_{\overline{d}-\overline{e}})$ *is an isomorphism onto its image.*

Proof. This proof is the same as the proof of [CF2, Theorem 2.3(ii)]. $\square$

LEMMA 5.3. *We have*

$$\phi_{\overline{e}}^{-1}\big(\overline{\Omega}_w(t)\big) = \pi^{-1}\big(\mathbb{P}^1 \times \overline{\Omega}_w(t)\big) \bigcup \tilde{\Omega}^{\overline{e}}_w(t),$$

$\tilde{\Omega}^{\overline{e}}_w(t)$ *being the locus inside* $\mathcal{U}_{\overline{e}}(t) := \pi^{-1}(\{t\} \times \mathcal{HQ}_{\overline{d}-\overline{e}})$, *where*

$$\operatorname{rank}\big(V_p \otimes \mathbb{O} \to K_q^*\big) \le r_w(q, p), \tag{5.5}$$

for all $p = 1, \ldots, n$, $q \in \{n_1, \ldots, n_k\}$.

Proof. See [CF2, Lemma 3.1]. $\square$

Following [CF2, Section 3], we now describe the locus $\tilde{\Omega}^{\overline{e}}_w(t)$ of Lemma 5.3. The analysis there can be applied in our case without any changes; the only reason for reproducing part of it here is to fix the somewhat elaborate notation needed.

Let $a := \operatorname{card}\{n_i - e_i \mid i = 1, \ldots, k\}$. Set $e_0 = e_{k+1} = 0$ and define a partition of $[0, k+1]$ as

$$i_0 = 0, \qquad i_j = \min\{i \mid n_i - e_i \ge n_{i_{j-1}} - e_{i_{j-1}} + 1\}, \quad \text{for } 1 \le j \le a, i_{a+1} = k+1.$$

Let $m_j = n_{i_j} - e_{i_j}$, for $j = 0, 1, \ldots, a$. By definition, on each of the intervals

$$\big[1, i_1 - 1\big], \big[i_1, i_2 - 1\big], \ldots, \big[i_{a-1}, i_a - 1\big], \big[i_a, k\big],$$

$n_i - e_i$ is constant and equal to $0, m_1, \ldots, m_a$, respectively. The corresponding bundles K_i^* are all isomorphic. Therefore, we can restrict the set of rank conditions (5.5), defining $\tilde{\Omega}^{\overline{e}}_w(t)$ inside $\mathcal{U}_{\overline{e}}(t)$, to

$$\operatorname{rank}\big(V_p \otimes \mathbb{O} \to K_q^*\big) \le r_w(q, p), \quad \text{for } 1 \le p \le n, q \in \big\{n_{i_1}, \ldots, n_{i_a}\big\}. \tag{5.6}$$

Moreover, we can further modify (5.6). Define recursively $\mathbf{r} := (r_{j,p})_{1 \le p \le n, 1 \le j \le a}$ as follows:

$$r_{1,p} = \min\big\{r_w(n_{i_1}, p), m_1\big\}, \quad 1 \le p \le n,$$

$$r_{j,p} = \min\big\{r_w\big(n_{i_j}, p\big), r_{j-1,p} + m_j - m_{j-1}\big\}, \quad \text{for } 1 \le p \le n, 2 \le j \le a.$$

LEMMA 5.4. *The conditions*

$$\operatorname{rank}\left(V_p \otimes \mathbb{O} \to K^*_{i_j}\right) \le r_{j,p}, \quad \textit{for } 1 \le p \le n, 1 \le j \le a \tag{5.7}$$

define the same degeneracy locus $\tilde{\Omega}^{\overline{e}}_w(t)$ *in* $\mathcal{U}_{\overline{e}}(t)$.

Proof. See [CF2, Construction 3.5]. □

LEMMA 5.5. (i) *There exists a unique permutation* $\tilde{w}^{\overline{e}} \in S_n$ *such that if* $\tilde{w}^{\overline{e}}(q) > \tilde{w}^{\overline{e}}(q+1)$, *then* $q \in \{m_1, \dots, m_a\}$ *and* $r_{j,p} = r_{\tilde{w}^{\overline{e}}}(m_j, p)$, *for all* $1 \le p \le n, 1 \le j \le a$.

(ii) *We have* $\ell(w) - \ell(\tilde{w}^{\overline{e}}) \le \sum_{i=1}^k e_i(n_{i+1} - n_i)$.

Proof. The following explicit construction of $\tilde{w}^{\overline{e}}$ is taken from [CF2, Lemma 3.6].

For each $j = 0, 1, \dots, a+1$, define sets $W_j(w)$ by

$$W_0(w) = \emptyset, \qquad W_j(w) = \left\{w(1), \dots, w\left(n_{i_j}\right)\right\}.$$

Also, we define sets $Z_j(w)$ and *ordered* sets $\tilde{Z}_j(w)$ such that

(a) card $Z_j(w) = n_{i_j} - m_{j-1}$,

(b) card $\tilde{Z}_j(w) = m_j - m_{j-1}$,

(c) $\tilde{Z}_j(w) \bigcap \tilde{Z}_{j'}(w) = \emptyset$, if $j \ne j'$,

(d) $\bigcup_{j=1}^{a+1} \tilde{Z}_j(w) = [1, n]$, by the following recursive procedure.

Let $Z_0(w) = \tilde{Z}_0(w) = \emptyset$. If $\tilde{Z}_i(w)$ was already defined for $i = 0, 1, \dots, j-1$, let

$$Z_j(w) = W_j(w) \setminus \left(\bigcup_{i=1}^{j-1} \tilde{Z}_i(w)\right).$$

Arrange $Z_j(w)$ in increasing order

$$Z_j(w) = \left\{z_{j,1} < \cdots < z_{j,n_{i_j} - m_{j-1}}\right\},$$

and set

$$\tilde{z}_{m_{j-1}+1} := z_{j,1}, \tilde{z}_{m_{j-1}+2} := z_{j,2}, \dots, \tilde{z}_{m_j} := z_{j,m_j - m_{j-1}},$$

$$\tilde{Z}_j(w) := \left\{\tilde{z}_{m_{j-1}+1} < \tilde{z}_{m_{j-1}+2} < \cdots < \tilde{z}_{m_j}\right\}.$$

Now define $\tilde{w}^{\overline{e}}(q) = \tilde{z}_q$, for all $1 \le q \le n$.

The estimate (ii) follows (cf. [CF2, Lemma 3.8]) by first noticing that the difference $\ell(w) - \ell(\tilde{w}^{\overline{e}})$ is maximized by the longest permutation w°, defined by $w^\circ(i) = n - n_j + i - n_{j-1}$, for all $n_{j-1} + 1 \le i \le n_j$ and $1 \le j \le k+1$. For this case, one computes directly that we have, in fact, the equality

$$\ell\left(w^\circ\right) - \ell\left(\left(\tilde{w^\circ}\right)^{\overline{e}}\right) = \sum_{i=1}^k e_i(n_{i+1} - n_i).$$

□

Remark 5.6. (i) In the terminology of [F1], Lemma 5.5(i) says that $\mathbf{r}$ is a *permissible* collection of rank numbers.

(ii) Let $\tilde{F}_{\overline{e}} := F(m_1, \ldots, m_a, V)$ be the partial flag variety corresponding to the m_i's. The sequence of quotient bundles

$$V \otimes \mathbb{O}_{\mathcal{U}_{\overline{e}}(t)} \twoheadrightarrow K^*_{n_{i_a}} \twoheadrightarrow \cdots \twoheadrightarrow K^*_{n_{i_1}}$$

is the pullback via a uniquely determined morphism $\psi_{\overline{e}}(t) : \mathcal{U}_{\overline{e}}(t) \to \tilde{F}_{\overline{e}}$ of the tautological sequence of quotient bundles on $\tilde{F}_{\overline{e}}$. By Lemmas 5.4 and 5.5(i), $\tilde{w}^{\overline{e}}$ defines a Schubert variety on $\tilde{F}_{\overline{e}}$, and we have $\tilde{\Omega}^{\overline{e}}_w(t) = \psi_{\overline{e}}(t)^{-1}(\Omega_{\tilde{w}^{\overline{e}}})$.

Finally, we spell out in more detail what the analysis in this section says for some special cases.

LEMMA 5.7. *Let $(e_1, \ldots, e_k)$ be a multi-index. Fix $1 \leq j \leq k$ and $1 \leq i \leq n_j$, and consider the cycle $\alpha_{i,j} := s_{n_j - i+1} \times \cdots \times s_{n_j}$. Then*

$$\tilde{\alpha}^{\overline{e}}_{i,j} = \begin{cases} \alpha_{i,j}, & \text{if } e_j = 0, \\ \alpha_{i,j} \cdot s_{n_j} \times \cdots \times s_{n_j - e_j + 1}, & \text{if } 1 \leq e_j < i, \\ \mathrm{id}, & \text{if } i \leq e_j. \end{cases}$$

In particular, $\ell(\alpha_{i,j}) - \ell(\tilde{\alpha}^{\overline{e}}_{i,j}) \leq e_j$, with equality if and only if $e_j \leq i$.

Proof. The proof is immediate from the construction of $\tilde{\alpha}^{\overline{e}}_{i,j}$ in Lemma 5.5. □

LEMMA 5.8 (Compare [CF2, Lemma 3.9]). *Assume in addition that $\overline{e} \neq (0, \ldots, 0)$. Then we have the following.*

(i) $\sum_{i=1}^k e_i(e_i - e_{i-1}) \geq e_j$, *for $1 \leq j \leq k$. In particular, $\sum_{i=1}^k e_i(e_i - e_{i-1}) \geq 1$.*

(ii) *There exists $1 \leq j \leq k$ such that $\sum_{i=1}^k e_i(e_i - e_{i-1}) = e_j$ if and only if there are integers $1 \leq h_1 < h_2 < \cdots < h_m \leq j \leq l_m < \cdots < l_2 < l_1 \leq k$ such that*

$$e_i = \begin{cases} 0, & \text{for } i \in [1, h_1 - 1] \cup [l_1 + 1, k], \\ 1, & \text{for } i \in [h_1, h_2 - 1] \cup [l_2 + 1, l_1], \\ 2, & \text{for } i \in [h_2, h_3 - 1] \cup [l_3 + 1, l_2], \\ \cdots & \\ m, & \text{for } i \in [h_m, l_m] \end{cases}$$

(in particular, $e_j = m$).

(iii) *Let $\overline{e}_{\mathbf{hl}}$ denote a multi-index as in (ii), and let $w \in S$ be any permutation. Let $\tilde{w}^{\overline{e}_{\mathbf{hl}}}$ be the permutation given by Lemma 5.5(i). Then*

$$\ell(w) - \ell\big(\tilde{w}^{\overline{e}_{\mathbf{hl}}}\big) = \sum_{i=1}^{k} e_i\big(n_{i+1} - n_i\big) = \sum_{c=1}^{m} \big(n_{l_c+1} - n_{h_c}\big)$$

if and only if for every $1 \leq i \leq m$ *we have*

$$w\big(n_{h_i}\big) > \max\big\{w\big(n_{h_i}+1\big), \dots, w\big(n_{l_i}\big), w\big(n_{l_i+1}\big)\big\}.$$

In this case, $\tilde{w}^{\bar{e}\mathbf{hl}} = w \cdot \gamma_{h_m,l_m} \cdot \gamma_{h_{m-1},l_{m-1}} \times \cdots \times \gamma_{h_1,l_1}$, *where* $\gamma_{h,l}$ *denotes the cyclic permutation* $s_{n_h} \times \cdots \times s_{n_{l+1}-1}$ *(cf. the paragraph before Theorem 3.1).*

Proof. (i) First, using the easy identity

$$\sum_{i=1}^{k} e_i(e_i - e_{i-1}) = \frac{1}{2}\Big[e_1^2 + (e_2-e_1)^2 + \cdots + (e_k - e_{k-1})^2 + e_k^2\Big]$$

and the change of variables $x_1 = e_1, x_2 = e_2 - e_1, \dots, x_k = e_k - e_{k-1}, x_{k+1} = e_k$, the inequality in (i) becomes

$$\sum_{i=1}^{j}\big(x_i^2 - 2x_i\big) + \sum_{i=j+1}^{k+1} x_i^2 \geq 0, \tag{5.8}$$

with the additional constraint $x_{k+1} = \sum_{i=1}^{k} x_i$. Now (5.8) is equivalent to

$$\sum_{i=1}^{j}(x_i - 1)^2 + \sum_{i=j+1}^{k+1} x_i^2 \geq j. \tag{5.9}$$

Changing variables again to $y_1 = x_1 - 1, \dots, y_j = x_j - 1, y_{j+1} = x_{j+1}, \dots, y_{k+1} = x_{k+1}$, we are reduced to proving

$$\sum_{i=1}^{k+1} y_i^2 \geq j, \tag{5.10}$$

subject to the constraint

$$y_{k+1} = j + \sum_{i=1}^{k} y_i. \tag{5.11}$$

Replacing j in (5.10) by $y_{k+1} - \sum_{i=1}^{k} y_i$, we need to show that

$$\big(y_{k+1}^2 - y_{k+1}\big) + \sum_{i=1}^{k}\big(y_i^2 + y_i\big) \geq 0. \tag{5.12}$$

Since y_i, for $1 \leq i \leq k+1$, are integers, both $y_{k+1}^2 - y_{k+1}$ and $y_i^2 + y_i$ are nonnegative and (5.12) follows.

(ii) We have equality in (5.12) if and only if y_{k+1} is equal to either zero or 1, and each y_i, for $1 \leq i \leq k$, is equal to either zero or -1. Using (5.11), we see that equality occurs in one of the following two cases:

- either $y_{k+1}=0$, exactly j among $y_1,\dots,y_k$ are equal to -1, and the rest are equal to zero;
- or $y_{k+1}=1$, exactly $j-1$ among $y_1,\dots,y_k$ are equal to -1, and the rest are equal to zero.

Changing the variables back to e_i, the statement in (ii) is obtained.

(iii) This follows by the construction of $\tilde{w}^{\overline{e}\mathbf{hl}}$. □

6. Proofs of the moving lemma, the quantum Pieri formula, and Theorem 3.14. This section is devoted to the proofs of Theorem 4.3(ii), Theorem 3.1, and Theorem 3.14. For this purpose we heavily use the structure of the boundary of $\mathcal{HQ}_{\overline{d}}$, described in the preceding section.

Throughout the rest of the paper, we work with suitable general translates of the Schubert varieties (or any other subvarieties) in F.

6.1. Proof of Theorem 4.3(ii) (Compare [CF2, Theorem 4.1]). We proceed by induction on $\overline{d}$. If $d_1=\cdots=d_k=0$, then $\mathcal{HQ}_{\overline{d}}=H_{\overline{d}}=F$, and there is nothing to prove. Assume that the statement is true for all $\overline{f}$ such that $f_i\le d_i$, for $1\le i\le k$, and $f_j<d_j$, for some $1\le j\le k$. Let $c:=\sum_{i=1}^N \ell(w_i)$. By Theorem 4.3(i) and Theorem 5.2(i), it is enough to show that

$$\operatorname{codim}_{\mathcal{HQ}_{\overline{d}}}\left(\bigcap_{i=1}^N \overline{\Omega}_{w_i}(t_i)\right)\bigcap\ \phi_{\overline{e}}(\mathcal{U}_{\overline{e}})>c, \tag{6.1}$$

for every multi-index $\overline{e}\ne(0,\dots,0)$, satisfying conditions (5.1) and (5.2). Using Theorem 5.2(ii) and Lemma 5.1, the inequality (6.1) follows if we prove that the codimension of $\bigcap_{i=1}^N \phi_{\overline{e}}^{-1}(\overline{\Omega}_{w_i}(t_i))$ in $\mathcal{U}_{\overline{e}}$ is greater than

$$c-(\dim\mathcal{HQ}_{\overline{d}}-\dim\mathcal{U}_{\overline{e}})=c+1-\sum_{i=1}^k e_i(n_{i+1}-n_i)-\sum_{i=1}^k e_i(e_i-e_{i-1}).$$

By Lemma 5.3, we have to prove the same estimate for the codimension of

$$\bigcap_{i=1}^N\left(\pi^{-1}\big(\mathbb{P}^1\times\overline{\Omega}_{w_i}(t_i)\big)\bigcup\tilde{\Omega}^{\overline{e}}_{w_i}(t_i)\right) \tag{6.2}$$

in $\mathcal{U}_{\overline{e}}$. Since the points $t_1,\dots,t_N$ are distinct, the only possibly nonempty intersections in (6.2) contain either no $\tilde{\Omega}^{\overline{e}}_{w_i}(t_i)$ term or only one such term. If there is no such term, the required inequality follows from the induction assumption on $\mathcal{HQ}_{\overline{d}-\overline{e}}$ and the fact that π is a smooth map. After possibly renumbering the points t_i, to finish the proof it suffices to show the estimate for

$$W\bigcap\tilde{\Omega}^{\overline{e}}_{w_N}(t_N)\subset\mathcal{U}_{\overline{e}}(t_N), \tag{6.3}$$

where

$$W := \bigcap_{i=1}^{N-1} \Big(\pi^{-1}\big(\{t_N\} \times \overline{\Omega}_{w_i}(t_i)\big)\Big)$$

and $\mathcal{U}_{\overline{e}}(t_N) = \pi^{-1}(\{t_N\} \times \mathcal{HQ}_{\overline{d}-\overline{e}})$. By the induction assumption, W has codimension $c-\ell(w_N)$ in $\mathcal{U}_{\overline{e}}(t_N)$, while by Remark 5.6(ii) and Kleiman's theorem on transversality of general translates, the intersection (6.3) has codimension $\ell(\tilde{w}_N^{\overline{e}})$ in W. The estimate follows now from Lemma 5.5(ii) and Lemma 5.8(i). □

6.2. Computing GW invariants via degenerations. For the proofs of quantum Pieri and of Theorem 3.14, we need to compute certain invariants $\langle \Omega_{w_1}, \dots, \Omega_{w_N}\rangle_{\overline{d}}$. The technique we use is to degenerate the intersection $\bigcap_{i=1}^N \overline{\Omega}_{w_i}(t_i)$ by allowing some of the points t_i to coincide. This procedure may lead to contributions supported on the boundary, which can be evaluated using the analysis in Section 5. In this subsection, we summarize some results of this type.

The following is an easy consequence of Proposition 4.8 and Theorem 4.3. For a proof, see, for instance, [Be, Lemma 2.5].

LEMMA 6.1. *Let Y_1, Y_2 be subvarieties in F such that* $\operatorname{codim} Y_1 + \operatorname{codim} Y_2 = \dim \mathcal{HQ}_{\overline{d}}$, *and let $t_1, t_2 \in \mathbb{P}^1$ be distinct points. Assume $\overline{d} \neq (0, \dots, 0)$. Then*

$$\int_{\mathcal{HQ}_{\overline{d}}} \Big[\overline{Y_1(t_1)}\Big] \cup \Big[\overline{Y_2(t_2)}\Big] = 0.$$

In particular, for any $v, w \in S$,

$$\big\langle \Omega_v, \Omega_w \big\rangle_{\overline{d}} = \begin{cases} 1, & \textit{if } \overline{d} = (0, \dots, 0) \textit{ and } v = \check{w}, \\ 0, & \textit{otherwise}. \end{cases}$$

LEMMA 6.2. *Let $\overline{d}$ be a multi-index, and let $v_1, \dots, v_N, w_1, \dots, w_M \in S$ satisfy $\sum_{i=1}^N \ell(v_i) + \sum_{j=1}^M \ell(w_j) = \dim H_{\overline{d}}$. Let $y, t_1, \dots, t_M \in \mathbb{P}^1$ be distinct points. Assume that the conclusion of Theorem 4.3(ii) holds for the intersection*

$$\overline{\Omega}_{v_1}(y) \bigcap \cdots \bigcap \overline{\Omega}_{v_N}(y) \bigcap \overline{\Omega}_{w_1}(t_1) \bigcap \cdots \bigcap \overline{\Omega}_{w_M}(t_M). \tag{6.4}$$

Let $Y := \Omega_{v_1} \bigcap \cdots \bigcap \Omega_{v_N} \subset F$. Then

$$\big\langle \Omega_{v_1}, \dots, \Omega_{v_N}, \Omega_{w_1}, \dots, \Omega_{w_M} \big\rangle_{\overline{d}} = \int_{\mathcal{HQ}_{\overline{d}}} \big[\overline{Y(y)}\big] \cup \big[\overline{\Omega}_{w_1}(t_1)\big] \cup \cdots \cup \Big[\overline{\Omega}_{w_N}(t_N)\Big].$$

Proof. By Theorem 4.3(i), the intersection

$$\Omega_{v_1}(y) \bigcap \cdots \bigcap \Omega_{v_N}(y) \bigcap \Omega_{w_1}(t_1) \bigcap \cdots \bigcap \Omega_{w_M}(t_M) \subset H_{\overline{d}}$$

has pure dimension zero, and by assumption it coincides with (6.4). On the other hand, we have

$$\Omega_{v_1}(y) \bigcap \cdots \bigcap \Omega_{v_N}(y) = ev^{-1}(Y) \bigcap \{y\} \times H_{\overline{d}}.$$

The lemma follows now from the definition of $\langle \cdots \rangle_{\overline{d}}$, Corollary 4.4, and Proposition 4.8. $\square$

PROPOSITION 6.3. *Let $\overline{d} \neq (0, \ldots, 0)$ be a multi-index; let $u, v, w \in S$ be such that $\ell(u) + \ell(v) + \ell(w) = \dim H_{\overline{d}}$; and let $y, t \in \mathbb{P}^1$ be distinct points. Denote*

$$Z := \overline{\Omega}_u(y) \bigcap \overline{\Omega}_v(y) \bigcap \overline{\Omega}_w(t) \subset \mathcal{HQ}_{\overline{d}}.$$

Assume that Z is either empty or purely zero-dimensional. Then

(i) *Z is contained in $\mathcal{B}_{\overline{d}}$;*

(ii) *$[Z] = [\overline{\Omega}_u(y)] \cup [\overline{\Omega}_v(y)] \cup [\overline{\Omega}_w(t)]$ is a cycle of degree $\langle \Omega_u, \Omega_v, \Omega_w \rangle_{\overline{d}}$.*

Proof. (i) Write $\Omega_u \bigcap \Omega_v = Y$ inside F. Let Z' be the (largest) subscheme of Z supported on $H_{\overline{d}}$. Then, as in the proof of Lemma 6.2, we have

$$Z' = Y(y) \bigcap \Omega_w(t);$$

hence, by Proposition 4.8,

$$\operatorname{card} Z' = \int_{\mathcal{HQ}_{\overline{d}}} \left[\overline{Y(y)}\right] \cup \left[\overline{\Omega}_w(t)\right].$$

By Lemma 6.1, Z' is empty.

(ii) Let $V := \overline{\Omega}_v(y) \bigcap \overline{\Omega}_w(t) \subset \mathcal{HQ}_{\overline{d}}$, and consider the trivial family $\mathbb{P}^1 \times V \subset \mathbb{P}^1 \times \mathcal{HQ}_{\overline{d}}$ over $\mathbb{P}^1$. Let $\rho : X \hookrightarrow \mathbb{P}^1 \times \mathcal{HQ}_{\overline{d}} \xrightarrow{\text{pr}} \mathbb{P}^1$ be the family whose fibre over $x \in \mathbb{P}^1$ is the generalized Schubert variety $\overline{\Omega}_u(x)$. It follows from Theorem 4.3 that ρ is a fibre bundle map (see, e.g., [Be, Corollary 2.4]), and in particular it is flat. Since the intersection $(\mathbb{P}^1 \times V) \bigcap X$ is obviously proper over $\mathbb{P}^1$, the proposition follows from [F2, Example 10.2.1]. $\square$

6.3. Proof of quantum Pieri. We formulate first an auxiliary lemma, for which we introduce some notation.

Let X be a scheme. Let V be an n-dimensional complex vector space, and let B_i, for $1 \leq i \leq k$, be vector bundles on X, of ranks b_i, respectively. Fix an integer $j \in \{1, 2, \ldots, k\}$. Assume that we are given a sequence of generically injective maps

$$B_1 \to \cdots \to B_{j-1} \to B_j \to B_{j+1} \to \cdots \to B_k \to V^* \otimes \mathcal{O}_X.$$

Moreover, assume that $B_i \to V^*$ is an injective map of bundles for $1 \leq i \leq j$. Fix $0 \leq e \leq b_j - b_{j-1}$, and let $\rho : G_e(B_j/B_{j-1}) \to X$ be the Grassmann bundle of e-dimensional quotients of B_j/B_{j-1}, with universal sequence

$$0 \to L \to \rho^*(B_j/B_{j-1}) \to Q \to 0.$$

Let K be the natural induced extension

$$0 \to \rho^* B_{j-1} \to K \to L \to 0;$$

that is, K is the kernel of $\rho^* B_j \to Q$.

Let $V_1 \subset \cdots \subset V_{n-1} \subset V$ be a fixed flag, and let $w \in S_n$ be a permutation such that if $w(q) > w(q+1)$, then $q \in \{b_1, \ldots, b_{j-1}, b_j - e, b_{j+1}, \ldots, b_k\}$. Denote by $\mathbf{D}_w$ the degeneracy locus on $G_e(B_j/B_{j-1})$ determined by

$$\text{(6.5)} \quad \operatorname{rank}\left(V_p \otimes \mathcal{O} \to (\rho^* B_q)^*\right) \le r_w(q, p), \quad \text{for } 1 \le p \le n, q \in \{b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_k\},$$

and

$$\text{(6.6)} \quad \operatorname{rank}\left(V_p \otimes \mathcal{O} \to K^*\right) \le r_w(b_j - e, p), \quad \text{for } 1 \le p \le n.$$

Define a permutation $\hat{w} \in S_n$ as follows:

- let $\{z_1 < z_2 < \cdots < z_{b_j - b_{j-1}}\}$ be the set $\{w(b_{j-1}+1), w(b_{j-1}+2), \ldots, w(b_j)\}$, ordered increasingly;
- if $q \notin \{b_{j-1}+1, b_{j-1}+2, \ldots, b_j\}$, set $\hat{w}(q) = w(q)$;
- for $1 \le i \le b_j - b_{j-1}$, set $\hat{w}(b_{j-1}+i) = z_i$.

LEMMA 6.4. (i) *The image $\rho(\mathbf{D}_w)$ is the degeneracy locus $\mathbf{D}_{\hat{w}}$ on X defined by*

$$\text{(6.7)} \quad \operatorname{rank}\left(V_p \otimes \mathcal{O} \to B_q^*\right) \le r_{\hat{w}}(q, p), \quad 1 \le p \le n, q \in \{b_1, \ldots, b_k\}.$$

(ii) *The restriction of ρ to $\mathbf{D}_w$ has positive dimensional fibres, unless*

$$\text{(6.8)} \quad w(b_{j-1}+1) > w(b_j),$$

in which case

$$\text{(6.9)} \quad \hat{w} = w \cdot \underbrace{s_{b_j - e} \times \cdots \times s_{b_{j-1}+1}} \cdot \underbrace{s_{b_j - e+1} \times \cdots \times s_{b_{j-1}+2}} \times \cdots \times \underbrace{s_{b_j - 1} \times \cdots \times s_{b_{j-1}+e}}.$$

If (6.8) holds and $\mathbf{D}_{\hat{w}}$ is irreducible, then ρ maps $\mathbf{D}_w$ birationally onto $\mathbf{D}_{\hat{w}}$.

Proof. First note that if (6.8) is satisfied, then (6.9) follows directly from the definition of $\hat{w}$.

By the construction of $\hat{w}$, we have $r_{\hat{w}}(b_j, p) = r_w(b_j, p)$, for all $1 \le p \le n$. Since $w(b_j) < w(b_j + 1)$ by assumption, it follows from [F1, Proposition 4.2] that by adding the conditions

$$\operatorname{rank}\left(V_p \otimes \mathcal{O} \to (\rho^* B_j)^*\right) \le r_w(b_j, p), \quad \text{for } 1 \le p \le n,$$

to (6.5) and (6.6), we obtain the *same* locus $\mathbf{D}_w$ on $G_e(B_j/B_{j-1})$. In other words, $\mathbf{D}_w$ is contained in $\rho^{-1}(\mathbf{D}_{\hat{w}})$. Consider the Grassmann bundle obtained by restriction:

$$\rho : \rho^{-1}(\mathbf{D}_{\hat{w}}) \to \mathbf{D}_{\hat{w}}.$$

It is not hard to see that $\mathbf{D}_w$ is a Schubert variety in this bundle, of positive relative dimension, unless (6.8) holds, in which case it intersects each fibre in a point. The lemma follows. $\square$

We now prove the following equivalent reformulation of Theorem 3.1.

THEOREM 3.1′. *The GW number $\langle \Omega_{\alpha_{i,j}}, \Omega_w, \Omega_v \rangle_{\overline{d}}$ vanishes, unless $\overline{d}$ is one of the multi-indices $\overline{e}_{\mathbf{hl}}$ of Lemma 5.8, such that $\ell(w \cdot \gamma_{\mathbf{hl}}) = \ell(w) - \sum_{c=1}^{m} (n_{l_c+1} - n_{h_c})$ and v is dual to one of the permutations $w'' \cdot \delta_{\mathbf{hl}}$, in which case it is equal to* 1.

Proof of Theorem 3.1′. The idea is to degenerate the corresponding intersection of generalized Schubert varieties, as in the previous subsection. Specifically, let $\overline{d}$ be any multi-index not identically zero, and let $v \in S$ be such that $c := i + \ell(w) + \ell(v) = \dim H_{\overline{d}}$. Let $y, t \in \mathbb{P}^1$ be distinct points.

CLAIM 6.5. *The intersection*

$$Z := \overline{\Omega}_{\alpha_{i,j}}(y) \bigcap \overline{\Omega}_w(y) \bigcap \overline{\Omega}_v(t) \tag{6.10}$$

is either empty or purely zero-dimensional.

Proof. Indeed, by Theorem 4.3(i), it is enough to show that the restriction of Z to $\mathcal{B}_{\overline{d}}$ is either empty or purely of dimension zero. As in the proof of Theorem 4.3(ii), this reduces to showing that the codimension in $\mathcal{U}_{\overline{e}}$ of

$$\Big(\phi_{\overline{e}}^{-1}\big(\overline{\Omega}_{\alpha_{i,j}}(y)\big)\Big) \bigcap \Big(\phi_{\overline{e}}^{-1}\big(\overline{\Omega}_w(y)\big)\Big) \bigcap \Big(\phi_{\overline{e}}^{-1}\big(\overline{\Omega}_v(t)\big)\Big) \tag{6.11}$$

is at least $c + 1 - \sum_{i=1}^{k} e_i(n_{i+1} - n_i) - \sum_{i=1}^{k} e_i(e_i - e_{i-1})$, for all multi-indices $\overline{e} \neq (0, \ldots, 0)$, satisfying (5.1) and (5.2). By Lemma 5.3, the terms in (6.11) can be rewritten as

$$\pi^{-1}\big(\mathbb{P}^1 \times \overline{\Omega}_{\alpha_{i,j}}(y)\big) \bigcup \tilde{\Omega}^{\overline{e}}_{\alpha_{i,j}}(y),$$
$$\pi^{-1}\big(\mathbb{P}^1 \times \overline{\Omega}_w(y)\big) \bigcup \tilde{\Omega}^{\overline{e}}_w(y),$$
$$\pi^{-1}\big(\mathbb{P}^1 \times \overline{\Omega}_v(t)\big) \bigcup \tilde{\Omega}^{\overline{e}}_v(t),$$

respectively. We have seen already in the proof of Theorem 4.3(ii) that the only possibly nonempty intersection is

$$\tilde{\Omega}^{\overline{e}}_{\alpha_{i,j}}(y) \bigcap \tilde{\Omega}^{\overline{e}}_w(y) \bigcap \pi^{-1}\big(\{y\} \times \overline{\Omega}_v(t)\big),$$

which lies inside $\mathcal{U}_{\overline{e}}(y)$. But we can rewrite this as

$$\psi_{\overline{e}}(y)^{-1}\Big(\Omega_{\tilde{\alpha}^{\overline{e}}_{i,j}}\Big) \bigcap \psi_{\overline{e}}(y)^{-1}\big(\Omega_{\tilde{w}^{\overline{e}}}\big) \bigcap \pi^{-1}\big(\{y\} \times \overline{\Omega}_v(t)\big), \tag{6.12}$$

where $\psi_{\overline{e}}(y) : \mathcal{U}_{\overline{e}}(y) \to \tilde{F}_{\overline{e}}$ is the morphism of Remark 5.6(ii). The codimension of (6.12) in $\mathcal{U}_{\overline{e}}(y)$ is

$$\ell\big(\tilde{\alpha}_{i,j}^{\overline{e}}\big)+\ell\big(\tilde{w}^{\overline{e}}\big)+\ell(v)$$

by Kleiman's transversality theorem; hence, its codimension in $\mathcal{U}_{\overline{e}}$ is

$$1+\ell\big(\tilde{\alpha}_{i,j}^{\overline{e}}\big)+\ell\big(\tilde{w}^{\overline{e}}\big)+\ell(v)=1+c-\Big(\ell\big(\alpha_{i,j}\big)-\ell\big(\tilde{\alpha}_{i,j}^{\overline{e}}\big)\Big)-\Big(\ell(w)-\ell\big(\tilde{w}^{\overline{e}}\big)\Big).$$

The required estimate follows now from Lemmas 5.5(ii), 5.7, and 5.8(i). □

By Claim 6.5 and Proposition 6.3, the GW invariant $\langle \Omega_{\alpha_{i,j}}, \Omega_w, \Omega_v\rangle_{\overline{d}}$ can be computed as the degree of $[Z]$ in the Chow ring of the hyperquot scheme. But we know even more! Namely, if Z is nonempty, all the inequalities we used for the codimension estimates in the proof of Claim 6.5 must in fact be equalities. By Lemma 5.8(ii) and (iii), this implies that Z is contained in the (disjoint!) union of "strata"

$$\bigcup_{\overline{e}_{\mathbf{hl}}} \phi_{\overline{e}_{\mathbf{hl}}}\big(\mathcal{U}_{\overline{e}_{\mathbf{hl}}}(y)\big),$$

where the union is over all $\overline{e}_{\mathbf{hl}}$, such that $\ell(w\cdot\gamma_{\mathbf{hl}}) = \ell(w) - \sum_{c=1}^{m}(n_{l_c+1}-n_{h_c})$. Moreover, for each $\overline{e}_{\mathbf{hl}}$, as above, the preimage $\phi_{\overline{e}_{\mathbf{hl}}}^{-1}(Z)$ is given by the intersection (6.12), with $\overline{e}$ replaced by $\overline{e}_{\mathbf{hl}}$.

At this point we need the following lemma.

LEMMA 6.6. *The intersection*

$$\psi_{\overline{e}_{\mathbf{hl}}}(y)^{-1}\Big(\Omega_{\tilde{\alpha}_{i,j}^{\overline{e}_{\mathbf{hl}}}}\Big)\bigcap \psi_{\overline{e}_{\mathbf{hl}}}(y)^{-1}\big(\Omega_{\tilde{w}^{\overline{e}_{\mathbf{hl}}}}\big)\bigcap \pi^{-1}\big(\{y\}\times\overline{\Omega}_v(t)\big)$$

is empty whenever $\overline{d}\neq\overline{e}_{\mathbf{hl}}$.

Granting this for a moment, we complete the proof of quantum Pieri. Recall that $\mathcal{U}_{\overline{e}_{\mathbf{hl}}}(y)$ can be realized as a succession of Grassmann bundles over an open subscheme $\mathcal{V}\subset\{y\}\times\mathcal{HQ}_{\overline{d}-\overline{e}_{\mathbf{hl}}}$ (cf. the proof of Lemma 5.1). Lemma 6.6 says that Z is empty, except possibly when $\overline{d}$ is one of the multi-indices $\overline{e}_{\mathbf{hl}}$ described above. In this case,

$$\{y\}\times\mathcal{HQ}_{\overline{d}-\overline{e}_{\mathbf{hl}}}=\mathcal{V}=\{y\}\times H_{\overline{d}-\overline{e}_{\mathbf{hl}}}=\{y\}\times F,$$

and $\mathcal{U}_{\overline{e}_{\mathbf{hl}}}(y)$ is *projective*. Moreover, the map $\phi_{\overline{e}_{\mathbf{hl}}}(y) : \mathcal{U}_{\overline{e}_{\mathbf{hl}}}(y)\to\mathcal{HQ}_{\overline{d}}$ is an embedding, by Theorem 5.2(ii). It follows that the degree of $[Z]$ in the Chow ring of $\mathcal{HQ}_{\overline{d}}$ is given by

$$\int_{\mathcal{U}_{\overline{e}_{\mathbf{hl}}}(y)} \psi_{\overline{e}_{\mathbf{hl}}}(y)^*\Big[\Omega_{\tilde{\alpha}_{i,j}^{\overline{e}_{\mathbf{hl}}}}\Big]\cup\psi_{\overline{e}_{\mathbf{hl}}}(y)^*\big[\Omega_{\tilde{w}^{\overline{e}_{\mathbf{hl}}}}\big]\cup\pi^*\big[\Omega_v\big]. \tag{6.13}$$

Applying the classical Pieri formula (Theorem 1.5) on $\tilde{F}_{\overline{e}_{\mathbf{hl}}}$, we can rewrite (6.13) as

$$\sum_{\tilde{w}^{\overline{e}_{\mathbf{hl}}}\xrightarrow{\tilde{\alpha}_{i,j}^{\overline{e}_{\mathbf{hl}}}} w''}\int_{\mathcal{U}_{\overline{e}_{\mathbf{hl}}}(y)} \psi_{\overline{e}_{\mathbf{hl}}}(y)^*\big[\Omega_{w''}\big]\cup\pi^*\big[\Omega_v\big].$$

The subscheme $\psi_{\bar{e}_{\mathbf{hl}}}(y)^{-1}(\Omega_{w''})$ is the degeneracy locus inside $\mathcal{U}_{\bar{e}_{\mathbf{hl}}}(y)$ determined by

$$\operatorname{rank}\big(V_p \otimes \mathcal{O} \to K_q^*\big) \le r_{w''}(q,p), \quad p = 1,\dots,n,\ q \in \{m_1,\dots,m_a\}. \tag{6.14}$$

By Kleiman's transversality theorem, we may assume that both $\psi_{\bar{e}_{\mathbf{hl}}}(y)^{-1}(\Omega_{w''})$ and the intersection $\psi_{\bar{e}_{\mathbf{hl}}}(y)^{-1}(\Omega_{w''}) \bigcap \pi^{-1}(\Omega_v)$ have the expected codimension. Hence,

$$\psi_{\bar{e}_{\mathbf{hl}}}(y)^*\big[\Omega_{w''}\big] \cup \pi^*\big[\Omega_v\big] = \big[\psi_{\bar{e}_{\mathbf{hl}}}(y)^{-1}\big(\Omega_{w''}\big) \bigcap \pi^{-1}\big(\Omega_v\big)\big]$$

in the Chow ring of $\mathcal{U}_{\bar{e}_{\mathbf{hl}}}(y)$. Recall that $\pi : \mathcal{U}_{\bar{e}_{\mathbf{hl}}}(y) \to F$ can be realized as a succession of Grassmann bundle projections (cf. the proof of Lemma 5.1). Applying Lemma 6.4(i) to each of these Grassmann bundles, starting from the top, we get that the image of $\psi_{\bar{e}_{\mathbf{hl}}}(y)^{-1}(\Omega_{w''})$ under the projection π is the Schubert variety $\Omega_{w'''} \subset F$, where w''' is the permutation (in S!) obtained from w'' by the successive applications of Lemma 6.4(i). By Lemma 6.4(ii), it follows that $\pi_*[\psi_{\bar{e}_{\mathbf{hl}}}(y)^{-1}(\Omega_{w''})] = 0$, unless the condition (6.8) is satisfied in every instance where we have used Lemma 6.4(i), in which case $\pi_*[\psi_{\bar{e}_{\mathbf{hl}}}(y)^{-1}(\Omega_{w''})] = [\Omega_{w'''}]$.

Moreover, if this happens, the permutation w''' is obtained from w'' by applying successively the recipe (6.9). Using the fact that the simple transpositions s_i and s_j commute whenever i and j are not consecutive integers, it follows easily that $w''' = w'' \cdot \delta_{\mathbf{hl}}$ and

$$\ell\big(w'''\big) = \ell\big(w'' \cdot \delta_{\mathbf{hl}}\big) = \ell\big(w''\big) - m - \sum_{c=1}^{m} \big(n_{l_c} - n_{h_c - 1}\big).$$

From the projection formula,

$$\int_F \pi_*\big[\psi_{\bar{e}_{\mathbf{hl}}}(y)^{-1}\big(\Omega_{w''}\big) \bigcap \pi^{-1}\big(\Omega_v\big)\big] = \int_F \big[\Omega_{w''\cdot\delta_{\mathbf{hl}}}\big] \cup \big[\Omega_v\big]. \tag{6.15}$$

By Theorem 1.2, the latter intersection number vanishes, unless v is the permutation in S dual to $w'' \cdot \delta_{\mathbf{hl}}$, in which case it is equal to 1. This implies that the same holds for the intersection number (6.13).

Summarizing, $\deg[Z]$ vanishes, unless all the conditions stated in Theorem 3.1$'$ are satisfied, in which case it is equal to 1, and moreover, we have seen that $\deg[Z] = \langle \Omega_{\alpha_{i,j}}, \Omega_w, \Omega_v \rangle_{\bar{d}}$. This completes the proof of quantum Pieri. □

Proof of Lemma 6.6. For simplicity, we omit **h** and **l** from the notation. We first recall the situation we are dealing with. There is a diagram

$$\begin{array}{ccc} \mathcal{U}_{\bar{e}}(y) & \xrightarrow{\psi_{\bar{e}(y)}} & \tilde{F}_{\bar{e}} \\ {\scriptstyle\pi}\big\downarrow & & \\ \mathcal{V} & & \end{array}$$

with $\{y\} \times H_{\bar{d}-\bar{e}} \subset \mathcal{V} \subset \{y\} \times \mathcal{HQ}_{\bar{d}-\bar{e}}$ and π a composition of Grassmann bundle projections. Let

$$W := \tilde{\Omega}^{\overline{e}}_{\alpha_{i,j}}(y) \bigcap \tilde{\Omega}^{\overline{e}}_{w}(y) = \psi_{\overline{e}}(y)^{-1}\Big(\Omega_{\tilde{\alpha}^{\overline{e}}_{i,j}}\Big) \bigcap \psi_{\overline{e}}(y)^{-1}\big(\Omega_{\tilde{w}^{\overline{e}}}\big).$$

We may assume that W is irreducible, of the expected codimension $\ell(\tilde{\alpha}^{\overline{e}}_{i,j}) + \ell(\tilde{w}^{\overline{e}})$, while the intersection $W \bigcap \pi^{-1}(\{y\} \times \overline{\Omega}_v(t))$ is a nonempty finite set consisting of reduced points and supported on $\pi^{-1}(\{y\} \times H_{\overline{d}-\overline{e}})$. It then follows that $\pi(W) \bigcap (\{y\} \times \Omega_v(t))$ is a nonempty zero-dimensional subscheme of $\{y\} \times H_{\overline{d}-\overline{e}}$. By Lemma 6.1, this would imply $\overline{d} = \overline{e}$ and therefore conclude the proof, if we can show that $\pi(W) \bigcap (\{y\} \times H_{\overline{d}-\overline{e}})$ is of the form $Y(y)$, for some $Y \subset F$. Set $Y := ev_y(\pi(W))$, where ev_y is the restriction of the evaluation map to $\{y\} \times H_{\overline{d}-\overline{e}}$. Then $\pi(W) \subset Y(y)$. To get the reverse inclusion, it suffices to show that if there exists a map $f : \mathbb{P}^1 \to F$ with $[f] \in \pi(W)$, then for every $g : \mathbb{P}^1 \to F$ such that $g(y) = f(y)$ we have $[g] \in \pi(W)$ as well. The map f is represented by a sequence of subbundles

$$S_1 \subset S_2 \subset \cdots \subset S_k \subset V^* \otimes \mathbb{O}_{\mathbb{P}^1}.$$

By assumption, there exists a point in $W \subset \mathcal{U}_{\overline{e}}(y)$, lying over $[f]$. This is equivalent to saying that for every $i \in \{1, \ldots, k\}$ there exist quotients

$$S_i(y) \twoheadrightarrow \mathbb{C}^{e_i} \tag{6.16}$$

of the fibres at y, together with compatible maps $\mathbb{C}^{e_i} \to \mathbb{C}^{e_{i+1}}$, and which satisfy the degeneracy conditions defining $\tilde{\Omega}^{\overline{e}}_{\alpha_{i,j}}(y)$ and $\tilde{\Omega}^{\overline{e}}_{w}(y)$. If g is another map and $g(y) = f(y)$, then the flag of fibres at y for the sequence of subbundles corresponding to g coincides with

$$S_1(y) \subset S_2(y) \subset \cdots \subset S_k(y) \subset V^* \otimes \mathbb{O}_y.$$

Hence, we can take the *same* quotients (6.16) to obtain a point in W lying over $[g]$. The lemma is proved. □

6.4. Proof of Theorem 3.14. By the relation (4.2), it suffices to show that $\langle \Omega_{w_1}, \ldots, \Omega_{w_N}, \Omega_v \rangle_{\overline{d}} = 0$, for every $v \in S$ and for every $\overline{d}$ not identically zero. We may assume that $\ell(v) + \sum_{j=1}^N \ell(w_j) = \dim H_{\overline{d}}$. Let $y, t \in \mathbb{P}^1$ be distinct points, and let

$$\Omega_{w_1} \bigcap \cdots \bigcap \Omega_{w_N} := Y \subset F.$$

By Lemma 5.7 and conditions (1) and (2) in Theorem 3.14, for every multi-index $\overline{e}$ we have the inequality

$$\sum_{m=1}^{N} \Big(\ell(w_m) - \ell\big(\tilde{w}^{\overline{e}}_m\big)\Big) \le \sum_{j=1}^{k} e_j\big(n_{j+1} - n_j\big).$$

Using this, codimension estimates similar to the ones in the proofs of Theorem 4.3(ii) and Claim 6.5 show that the intersection

$$\overline{\Omega}_{w_1}(y) \bigcap \cdots \bigcap \overline{\Omega}_{w_N}(y) \bigcap \overline{\Omega}_v(t)$$

misses the boundary of $\mathcal{HQ}_{\overline{d}}$. Therefore we can apply Lemma 6.2 to conclude that

$$\left\langle \Omega_{w_1}, \ldots, \Omega_{w_M}, \Omega_v \right\rangle_{\overline{d}} = \int_{\mathcal{HQ}_{\overline{d}}} \left[\overline{Y(y)}\right] \cup \left[\overline{\Omega}_v(t)\right].$$

By Lemma 6.1, all such intersection numbers vanish whenever $\overline{d} \neq (0, \ldots, 0)$. □

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