QUANTUM DOUBLE SCHUBERT POLYNOMIALS

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December 11, 1996

Fomin, Gelfand, and Postnikov [2] have defined *quantum Schubert polynomials* $\mathfrak{S}^q_w(x)$, and proved that they represent the corresponding Schubert varieties in the quantum cohomology ring of a flag manifold. For a permutation w in S_n, $\mathfrak{S}^q_w(x)$ is a polynomial in variables $x_1, \ldots, x_{n-1}$ and $q_1, \ldots, q_{n-2}$, homogeneous of degree equal to the length $\ell(w)$ of w, where each x_i has degree 1 and each q_i has degree 2. These quantum Schubert polynomials specialize to the Schubert polynomials $\mathfrak{S}_w(x)$ of Lascoux and Schützenberger when each q_i is set equal to 0.

Quantum Schubert polynomials are based on quantum analogues $E_i(x_1, \ldots, x_p)$ of the elementary symmetric polynomials $e_i(x_1, \ldots, x_p)$. These polynomials can be defined by taking a diagram with p vertices labeled by $x_1, \ldots, x_p$, with edges between vertex x_j and x_{j+1} labeled q_j. Then $E_i(x_1, \ldots, x_p)$ is the sum of all monomials in $x_1, \ldots, x_p, q_1, \ldots, q_{p-1}$ obtained by choosing vertices and edges that cover exactly i vertices without overlap, and multiplying the corresponding elements together. For any $J = (j_1, \ldots, j_{n-1})$ with $0 \le j_p \le p$ for all p, let

$$e_J(x) = \prod_{p=1}^{n} e_{j_p}(x_1, \ldots, x_p), \qquad E_J(x) = \prod_{p=1}^{n} E_{j_p}(x_1, \ldots, x_p). \tag{1}$$

Write $\mathfrak{S}_w(x) = \sum a_J\, e_J(x)$ for some integers a_J (this is always possible, for unique such integers). Then the quantum Schubert polynomial is defined [2] by

$$\mathfrak{S}^q_w(x) = \sum a_J\, E_J(x). \tag{2}$$

The classical Schubert polynomials of Lascoux and Schützenberger have the following simple description. Write $w = w_\circ \cdot s_{i_1} \cdot \ldots \cdot s_{i_\ell}$, where $\ell = \ell(w_\circ) - \ell(w)$ and $w_\circ = n\ n-1\ \ldots\ 1$ is the permutation in S_n of longest length. Then

$$\mathfrak{S}_w(x) = \partial^x_{i_\ell} \circ \cdots \circ \partial^x_{i_1}(\mathfrak{S}_{w_\circ}(x)), \tag{3}$$

where $\mathfrak{S}_{w_\circ}(x) = x_1^{n-1} x_2^{n-2} \ldots x_{n-1}$ and ∂^x_i is the operator that takes a polynomial P to $(P - s^x_i)(P)/(x_i - x_{i+1})$, with s^x_i the operator that interchanges x_i and x_{i+1}.

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There do not appear to be any operators in polynomials in the x's and q's that would give an analogue of (3) for the quantum Schubert polynomials. Nevertheless, our goal in this note is to prove an equally simple formula for the quantum Schubert polynomials.

The key to this is the observation that the classical Schubert polynomials are specializations of *double Schubert polynomials*, which have the following definition[1]: with w and $w_\circ$ related as above,

$$\mathfrak{S}_w(x,y) = \partial^x_{i_\ell} \circ \cdots \circ \partial^x_{i_1}(\mathfrak{S}_{w_\circ}(x,y)), \tag{4}$$

where $\mathfrak{S}_{w_\circ}(x,y) = \prod_{i+j\leq n}(x_i + y_j)$. Note that $\mathfrak{S}_w(x,0) = \mathfrak{S}_w(x)$. These double Schubert polynomials, however, have an important symmetry property, that $\mathfrak{S}_w(x,y) = \mathfrak{S}_{w^{-1}}(y,x)$ (cf. [3], (6.4)). This means that if we write $w = s_{j_\ell} \cdot \ldots \cdot s_{j_1} \cdot w_\circ$, with $\ell = \ell(w_\circ) - \ell(w)$, then

$$\mathfrak{S}_w(x,y) = \partial^y_{j_\ell} \circ \cdots \circ \partial^y_{j_1}(\mathfrak{S}_{w_\circ}(x,y)). \tag{5}$$

We will define *quantum double Schubert polynomials* $\mathfrak{S}_w(x,y,q)$ in variables $x_1,\ldots,x_{n-1},y_1,\ldots,y_{n-1},q_1,\ldots,q_{n-2}$, that will simultaneously generalize the double Schubert polynomials and the quantum Schubert polynomials, and which are defined by a formula like (5). Our definition is, with w and $w_\circ$ related as in (5),

$$\mathfrak{S}_w(x,y,q) = \partial^y_{j_\ell} \circ \cdots \circ \partial^y_{j_1}(\mathfrak{S}_{w_\circ}(x,y,q)), \tag{6}$$

with

$$\mathfrak{S}_{w_\circ}(x,y,q) = \prod_{p=1}^{n-1} E_p(x_1 + y_{n-p}, \ldots, x_p + y_{n-p}). \tag{7}$$

When each q variable is set equal to 0, $E_p(x_1 + y_{n-p}, \ldots, x_p + y_{n-p})$ becomes $\prod_{i=1}^p (x_i + y_{n-p})$, from which we see that

$$\mathfrak{S}_w(x,y,0) = \mathfrak{S}_w(x,y). \tag{8}$$

Our goal is the

Theorem. $\mathfrak{S}^q_w(x,0,q) = \mathfrak{S}^q_w(x)$.

Proof : It suffices to prove the following claim: each $\mathfrak{S}_w(x,y,q)$ can be written in the form $\sum a_J(y)\, E_J(x)$ for some polynomials $a_J(y)$ in $\mathbb{Z}[y_1,\ldots,y_{n-1}]$. Granting this claim, we have an equation

$$\mathfrak{S}_w(x,0,q) = \sum a_J(0)\, E_J(x).$$

[1] The double Schubert polynomials of Lascoux and Schützenberger are obtained from those defined here by replacing each y_i by $-y_i$; this change is made only to avoid excessive signs in our formulas.

Since $\mathfrak{S}_w(x,0,0) = \sum a_J(0)\, e_J(x)$, and by (4), $\mathfrak{S}_w(x,0,0) = \mathfrak{S}_w(x)$, the theorem follows.

To prove the claim, let M be the $\mathbb{Z}[y]-$ submodule of the polynomial ring $\mathbb{Z}[x,y,q]$ generated by the $E_J(x)$, as J varies over index sets in (1). It is obvious that M is preserved by the operators ∂_i^y, so it suffices to prove that $\mathfrak{S}_{w_\circ}(x,y,q)$ is in M. But this follows from the easy identity

$$E_p(x_1 + y_{n-p}, \ldots, x_p + y_{n-p}) = \sum_{i=0}^{p} E_i(x_1,\ldots,x_p) y_{n-p}^{n-i},$$

and our definition (7) of $\mathfrak{S}_{w_\circ}(x,y,q)$. $\square$

A similar argument shows that any $\mathfrak{S}_w(x,y,q)$ can be written in the form $\sum_{u\in S_n} T_u(y)\mathfrak{S}_u^q(x)$. Setting the q's equal to 0 and using [3], (6.3), one finds

$$\mathfrak{S}_w(x,y,q) = \sum_{\substack{v^{-1}u=w \\ \ell(u)+\ell(v)=\ell(w)}} \mathfrak{S}_v(y)\mathfrak{S}_u^q(x). \tag{9}$$

We conclude with a table of the quantum double Schubert polynomials for S_3:

$$\mathfrak{S}_{3\,2\,1}(x,y,q) = (x_1+y_2)((x_1+y_1)(x_2+y_1)+q_1)$$

$$\mathfrak{S}_{2\,3\,1}(x,y,q) = (x_1+y_1)(x_2+y_1)+q_1 \qquad \mathfrak{S}_{3\,1\,2}(x,y,q) = (x_1+y_1)(x_1+y_2)-q_1$$

$$\mathfrak{S}_{1\,3\,2}(x,y,q) = x_1+x_2+y_1+y_2 \qquad \mathfrak{S}_{2\,1\,3}(x,y,q) = x_1+y_1$$

$$\mathfrak{S}_{1\,2\,3}(x,y,q) = 1$$

In another paper we plan to show how these quantum double Schubert polynomials correspond to degeneracy loci on the hyperquot schemes studied in [1]

Acknowledgements. The second author thanks Piotr Pragacz for a discussion speculating about the possibility of defining quantum Schubert polynomials by means of operators. Both authors thank the Mittag-Leffler Institute for providing the setting where this research took place. The first author was supported by a Mittag-Leffler Institute postdoctoral fellowship. The second author was partially supported by a Tage Erlander Guest Professorship and the National Science Foundation.

As this note was being corrected, we discovered the preprint *"Quantum double Schubert polynomials, quantum Schubert polynomials and Vafa-Intriligator formula"* by A.N. Kirillov and T. Maeno, which contains the same results and more, but from a different point of view.

References

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