Quantum multiplication of Schur polynomials

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1 Introduction

The classical Giambelli formula writes the class of a Schubert variety in a Grassmannian as a Schur polynomial, and rules of Pieri and Littlewood-Richardson give explicit formulas for multiplying these classes. The quantum cohomology ring of a Grassmannian is known, together with formulas for the classes of Schubert varieties [3]. Our aim here is to present and discuss formulas for their quantum multiplication. Specifically, we give a formula expressing the *quantum Littlewood-Richardson numbers* (i.e., the coefficients that appear when expanding the quantum product of two Schubert classes in the basis of Schubert classes) as a sum of signed classical Littlewood-Richardson coefficients. We also give a combinatorial formula for the *quantum Kostka numbers*, which are the coefficients appearing in an iterated Pieri quantum product. Both the results and their proofs involve adjoining (or removing) rim hooks to the Young diagrams of partitions corresponding to Schubert varieties.

Fix positive integers k and l, and set $n = k + l$. Let $X := Gr(l, n)$ be the Grassmannian of l-planes in $\mathbf{C}^n$. Let $\sigma_1, \ldots, \sigma_k$ be indeterminates, with σ_i

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of degree i. Define polynomials $Y_r = Y_r(\sigma_1, \ldots, \sigma_k)$ by the formulas

$$Y_r = \det(\sigma_{1+j-i})_{1 \le i,j \le r}. \tag{1}$$

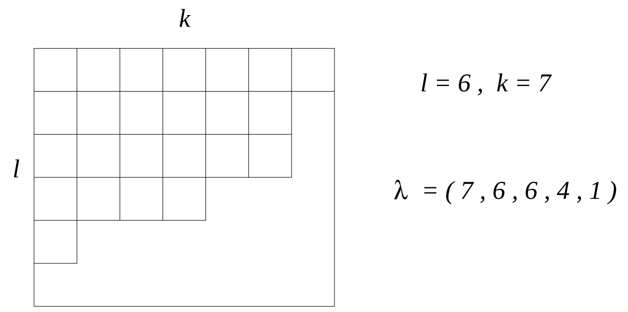
In the classical cohomology ring $H^*(X)$, if σ_i maps to the i^{th} Chern class of the universal quotient bundle, then $(-1)^r Y_r$ maps to the r^{th} Chern class of the universal subbundle. This leads to one of the standard presentations of the classical cohomology ring of the Grassmannian:

$$H^*(X) = \mathbf{Z}[\sigma_1, \ldots, \sigma_k]/(Y_{l+1}, \ldots, Y_n). \tag{2}$$

The (small) quantum cohomology ring $QH^*(X)$ is an algebra over $\mathbf{Z}[q]$, with q a variable of degree n. It has the presentation ([13], [3], [7])

$$QH^*(X) = \mathbf{Z}[q, \sigma_1, \ldots, \sigma_k]/(Y_{l+1}, \ldots, Y_{n-1}, Y_n + (-1)^k q). \tag{3}$$

We identify a partition λ with its Young diagram; we write $\lambda \subset l \times k$ for a partition $\lambda = (\lambda_1 \ge \ldots \ge \lambda_l \ge 0)$ with $\lambda_1 \le k$, i.e., for a partition whose Young diagram fits inside an $l \times k$ rectangle.



There is a Schubert variety $\Omega_\lambda \subset X$ for each $\lambda \subset l \times k$; Ω_λ consists of l-dimensional subspaces of $\mathbf{C}^n$ such that $\dim(L \cap \mathbf{C}^{k+i-\lambda_i}) \ge i$, for $1 \le i \le l$, where $\mathbf{C}^1 \subset \mathbf{C}^2 \subset \ldots \subset \mathbf{C}^n$ is a fixed flag. The classical Giambelli formula says that the class of Ω_λ in $H^*(X)$ is σ_λ, where

$$\sigma_\lambda = \det(\sigma_{\lambda_i+j-i})_{1 \le i,j \le l}. \tag{4}$$

These classes σ_λ give a $\mathbf{Z}$-basis for $H^*(X)$. Their product in this ring can therefore be written

$$\sigma_\lambda \cdot \sigma_\mu = \sum N_{\lambda\mu}^{\nu} \sigma_\nu. \tag{5}$$

Here λ, μ and ν are partitions inside the $l \times k$ rectangle, with sizes $|\nu| = |\lambda| + |\mu|$. That the coefficients $N_{\lambda\mu}^{\nu}$ are nonnegative follows from the fact that the Schubert varieties can be translated by the action of $GL_n(\mathbf{C})$ so that they meet properly. In fact, Grassmannians are one class of varieties for which explicit closed expressions are known for multiplying Schubert classes, which also show the coefficients to be nonnegative. The formula for multiplication is known as the *Littlewood-Richardson rule*, and states that $N_{\lambda\mu}^{\nu}$ is the number of tableaux on the skew shape ν/λ of content μ, whose word is a reverse lattice word. Here ν/λ is the complement of the Young diagram of λ in the Young diagram of ν. A *tableau* of *content* $\mu = (\mu_1, \ldots, \mu_l)$ is a numbering of the boxes of ν/λ with μ_1 1's, μ_2 2's, $\ldots$, up to μ_l l's, which are weakly increasing across rows, and strictly increasing down columns (these are often called *semistandard tableaux*). The *word* of a tableau is the list of its entries, read from left to right in rows, from bottom to top. A word is *reverse lattice* if, from any point in it to the end, there are at least as many 1's as there are 2's, at least as many 2's as 3's, and so on. An example with $\lambda = (2, 1)$, $\nu = (3, 3, 3, 1)$, $\mu = (3, 2, 2)$ is shown in the figure below.

$$\begin{array}{|c|c|c|}
\hline
 & & 1 \\
\hline
 & 1 & 2 \\
\hline
1 & 2 & 3 \\
\hline
3 \\
\cline{1-1}
\end{array}
\qquad\qquad
\begin{array}{|c|c|c|}
\hline
 & & 1 \\
\hline
 & 1 & 2 \\
\hline
1 & 3 & 3 \\
\hline
2 \\
\cline{1-1}
\end{array}$$

The word 3123121 of the first tableau is a reverse lattice word, while the word 2133121 of the second is not.

The rule follows from the corresponding identity for Schur functions

$$s_\lambda \cdot s_\mu = \sum N_{\lambda\mu}^{\nu} s_\nu, \tag{6}$$

where $s_\lambda = \det(s_{\lambda_i + j - i})$, with $s_1, s_2, \ldots$ indeterminates. In this case there is no need to restrict to partitions inside any rectangle; the map

$$s_i \longmapsto \sigma_i \in H^*(X)$$

takes s_λ to 0 if λ is not contained in the $l \times k$ rectangle (see [12], [6]).

The Pieri formula is a special case:

$$s_p \cdot s_\lambda = \sum s_\nu, \tag{7}$$

the sum over all ν obtained from λ by adding p boxes, with no two in the same column. Repeated, it gives a formula

$$s_{\mu_1} \cdot \ldots \cdot s_{\mu_r} \cdot s_\lambda = \sum K^\nu_{\lambda\mu} s_\nu. \tag{8}$$

Here $\mu_1, \ldots, \mu_r$ are nonnegative integers, the sum is over all ν with $|\nu| = |\lambda| + \sum_{i=1}^r \mu_i$, and $K^\nu_{\lambda\mu}$ is the *Kostka number*: the number of tableaux on ν/λ with content μ. The usual Kostka number $K_{\nu\mu}$ is $K^\nu_{\lambda\mu}$, where $\lambda = \emptyset$ and μ is a partition. It is a consequence of (8) that $K^\nu_{\lambda\mu}$ is independent of the order of terms in $(\mu_1, \ldots, \mu_r)$.

We turn now to the quantum analogues of these formulas. We identify $QH^*(X)$ with the ring presented in (3), and we regard $\sigma_1, \ldots, \sigma_k$ as elements of this ring. For any partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, we let

$$\sigma_\lambda = \det(\sigma_{\lambda_i + j - i})_{1 \le i,j \le r},$$

an element of $QH^*(X)$. The classes σ_λ, for $\lambda \subset l \times k$ form a $\mathbf{Z}[q]$ basis for $QH^*(X)$. A Schubert variety Ω_λ determines a class in $QH^*(X)$. It is a result of [3] that this class is still given by σ_λ; unlike other flag varieties (see [4], [8]), there are no correction terms involving the variable q.

It follows that in $QH^*(X)$ there are formulas

$$\sigma_\lambda \cdot \sigma_\mu = \sum q^m N^\nu_{\lambda\mu}(l,k) \sigma_\nu, \tag{9}$$

the sum over $m \ge 0$ and $\nu \subset l \times k$, but with $|\nu| = |\lambda| + |\mu| - mn$. The coefficients $N^\nu_{\lambda\mu}(l,k)$ are again nonnegative, this time for "quantum geometric" reasons: $N^\nu_{\lambda\mu}(l,k)$ is the number (properly counted) of rational curves of degree m that meet general Schubert varieties Ω_λ, Ω_μ, and $\Omega_{\nu^\vee}$, where $\nu^\vee = (k - \nu_l, \ldots, k - \nu_1)$ (cf. [3], [7]). The coefficients $N^\nu_{\lambda\mu}(l,k)$ are uniquely determined from the presentation (3) of the ring $QH^*(X)$. Similarly, for $0 \le \mu_1, \ldots, \mu_r \le k$ and $\lambda \subset l \times k$,

$$\sigma_{\mu_1} \cdot \ldots \cdot \sigma_{\mu_r} \cdot \sigma_\lambda = \sum q^m K^\nu_{\lambda\mu}(l,k) \sigma_\nu, \tag{10}$$

the sum over $m \ge 0$, $\nu \subset l \times k$, with $|\nu| = |\lambda| + \sum_{i=1}^r \mu_i - mn$. Our goal in this paper is to give explicit formulas for these *quantum Littlewood-Richardson numbers* $N^\nu_{\lambda\mu}(l,k)$ and *quantum Kostka numbers* $K^\nu_{\lambda\mu}(l,k)$. Note that they depend on our given numbers l and k, as well as on λ, μ and ν. Our result is most satisfying for the quantum Kostka numbers: we show that they are the number of tableaux satisfying a certain condition. Our algorithm for the quantum Littlewood-Richardson numbers is efficient, but, since it involves signs, it does not show them to be nonnegative.

If $\lambda_1 > k$, it follows from the definition that $\sigma_\lambda = 0$. If $\lambda_{l+1} > 0$, however, σ_λ need not vanish. Our basic algorithm gives a formula for all such polynomials σ_λ. This can be expressed in terms of removing n-rim hooks from λ (see Section 2).

Although quantum cohomology is not functorial, isomorphic varieties do have isomorphic quantum cohomology rings. The natural isomorphism of $QH^*(Gr(l,n))$ with $QH^*(Gr(k,n))$ leads to interesting dual versions of our results, which are discussed in Section 4.

In the final Section 5, we sketch the relation between the quantum cohomology ring and the Verlinde (or fusion) algebra, which was pointed out to us by C. Woodward. In this section we describe the proofs of Woodward and S. Agnihotri that products of Schubert classes can never vanish in the quantum cohomology ring.

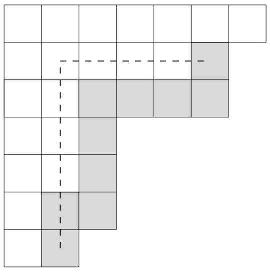
The content of this paper is algebraic and combinatorial; quantum cohomology is used only for motivation, and for the nonnegativity of the quantum Littlewood-Richardson numbers.

This work was carried out in the stimulating atmosphere of the Mittag-Leffler Institute. We thank F. Sottile and C. Greene for help with the literature on rim hooks, and C. Woodward and S. Agnihotri for the material in Section 5. The second named author was supported by a Mittag-Leffler Institute postdoctoral fellowship. The third author was supported by a Tage Erlander Guest Professorship, and the National Science Foundation.

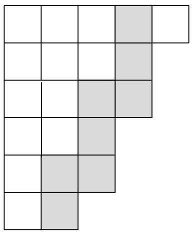
2 The rim hook algorithm

There is a *rim hook* of a partition λ corresponding to each of its boxes. It is an *n-rim hook* if it contains exactly n boxes (which is the same number of boxes as the hook of the corresponding box).

The figure below shows an n-rim hook, with $n = 10$.



Let λ be a partition with $\lambda_1 \leq k$, and let $\widetilde{\lambda}_i$ be the number of boxes in the i^{th} column of λ; i.e., the conjugate partition $\widetilde{\lambda}$ is $(\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_k)$. One can start at the bottom of any column and move upward and to the right until one has counted off n boxes along the rim. This will be an n-rim hook, unless this process ends in a box directly to the left of the last box in some column. We call this an *illegal n-rim.*



illegal 8-rim

If this process starts in column r and ends in column s, the column lengths of the shape that remains are

$$(\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{r-1}, \widetilde{\lambda}_{r+1}-1, \widetilde{\lambda}_{r+2}-1, \ldots, \widetilde{\lambda}_s-1, \widetilde{\lambda}_r-r+s-n, \widetilde{\lambda}_{s+1}, \ldots, \widetilde{\lambda}_k). \tag{11}$$

The n-rim is illegal exactly when $\widetilde{\lambda}_r - r + s - n = \widetilde{\lambda}_{s+1} - 1$, or

$$\widetilde{\lambda}_r - r - n = \widetilde{\lambda}_{s+1} - (s+1). \tag{12}$$

It is also possible that there is no n-rim starting in column r, which happens if $\widetilde{\lambda}_r + \lambda_1 - r < n$; so if $\widetilde{\lambda}_1 + \lambda_1 \leq n$, then λ contains no n-rim.

Otherwise s is the unique integer $\geq r$ with

$$\widetilde{\lambda}_s - s > \widetilde{\lambda}_r - r - n > \widetilde{\lambda}_{s+1} - (s+1). \tag{13}$$

In this case (11) is a partition $\widetilde{\mu}$, conjugate to the partition μ obtained after removing such an n-rim hook from λ.

The *width* of the n-rim hook is the number $s-r+1$ of columns it occupies.

For positive integers k, l, set $n = k + l$, and set

$$\Lambda(l,k) = \mathbf{Z}[q, \sigma_1, \ldots, \sigma_k]/(Y_{l+1}(\sigma), \ldots, Y_{n-1}(\sigma), Y_n(\sigma) + (-1)^k q) \ ,$$

where $Y_p(\sigma) = \det(\sigma_{1+j-i})_{1 \le i,j \le p}$. For any partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, let σ_λ be the image of $\det(\sigma_{\lambda_i+j-i})_{1\le i,j\le r}$ in $\Lambda(l,k)$.

Main Lemma *Let λ be any partition.*

(A) *If λ contains an illegal n-rim, or if $\lambda_{l+1} > 0$ and λ contains no n-rim, or if $\lambda_1 > k$, then $\sigma_\lambda = 0$.*

(B) *If μ is the result of removing an n-rim hook from λ, then*

$$\sigma_\lambda = (-1)^{k-w} q \sigma_\mu,$$

where w is the width of the n-rim hook removed.

Proof: Let $y_r = \sigma_{(1^r)}$ for all integers $r \ge 1$, with $y_0 = 1$ and $y_i = 0$ for $i < 0$; y_i is the image of Y_i in $\Lambda(l,k)$. We know by (1) and (3) that $y_i = 0$ for $l < i < n$, and $y_n = (-1)^{k-1} q$. By expanding the determinant expression for y_r along the top row, we have

$$y_r = \sigma_1 y_{r-1} - \sigma_2 y_{r-2} + \ldots + (-1)^{k-1} \sigma_k y_{r-k}.$$

From this it follows by induction that

$$y_{mn+j} = (-1)^{m(k-1)} q^m y_j \qquad \text{for} \quad 0 \le j \le n-1, \ m \ge 0. \tag{14}$$

For any sequence $m = (m_1, \ldots, m_k)$ of integers, set

$$\tau_m = \det(y_{m_i+j-i})_{1\le i,j\le k}. \tag{15}$$

If $\lambda_1 > k$, the vanishing of σ_λ follows directly from its definition. If λ is a partition with $\lambda_1 \le k$, it is a basic identity for symmetric polynomials (cf. [12], (2.9′)) that

$$\sigma_\lambda = \tau_{\widetilde{\lambda}} \tag{16}$$

If $1 \le r < s \le k$ and the r^{th} row of the matrix $(y_{m_i+j-i})_{1\le i,j\le k}$ is successively interchanged with the $(r+1)^{\text{st}}$ row, then the $(r+2)^{\text{nd}}$ row, and so on to the s^{th} row, one finds

$$\tau_m = (-1)^{s-r}\tau_{m'} \ , \tag{17}$$

where $m' = (m_1, \dots, m_{r-1}, m_{r+1}-1, \dots, m_s-1, m_r-r+s, m_{s+1}, \dots, m_k)$.

Set $m = \widetilde{\lambda}$. If $\lambda_{l+1} > 0$ and λ contains no n-rim, we have $m_1 > l$, $m_1 + \lambda_1 \le n$. Then $\sigma_\lambda = \det(y_{m_i+j-i})_{1\le i,j\le \lambda_1}$, which vanishes since its top row is zero.

Suppose now that the n-rim starting in column r of λ ends in column s. If this rim is illegal, it follows from (11) that $(y_{m'_i+j-i})$ has two identical rows, so $\sigma_\lambda = \tau_m = 0$. This completes the proof of (A).

If $\widetilde{\lambda}_r \ge l$ for some r between 1 and k, applying (14) to the entries of row s of $(y_{m'_i+j-i})$ yields

$$\tau_{m'} = (-1)^{k-1}q\tau_{m''} \ , \tag{18}$$

where $m'' = (m'_1, \dots, m'_{s-1}, m'_s - n, m'_{s+1}, \dots, m'_k)$. If the n-rim starting in column r of λ is a rim hook, and it ends in column s, then by (17) and (18)

$$\tau_m = (-1)^{k-1+s-r}q\tau_{m'''} \ ,$$

where $m''' = (m_1, \dots, m_{r-1}, m_{r+1}-1, \dots, m_s-1, m_r-r+s-n, m_{s+1}, \dots, m_k)$. Part (B) of the lemma follows from this, using (11) and (16). □

Given any partition λ with $\lambda_1 \le k$, one can succesively apply the lemma, removing n-rim hooks until one arrives at a partition μ which has no n-rim hooks. If $\mu \subset l \times k$, and $R_1, \dots, R_m$ are the rim hooks removed, then

$$\sigma_\lambda = \varepsilon(\lambda/\mu)q^m\sigma_\mu, \tag{19}$$

where $\varepsilon(\lambda/\mu) = (-1)^{\sum(k-\text{width}(R_i))}$. If μ is not contained in the $l \times k$ rectangle, then

$$\sigma_\lambda = 0. \tag{20}$$

One sees from (19) that if $\mu \subset l \times k$, then μ is independent of the choice of the m rim hooks removed, as is $\sum \text{width}(R_i) \bmod(2)$. This is a general fact: μ is the *n-core* of λ, which is uniquely determined by λ and n ([10], §2.7, cf. [12], §1).

Consider the specialization of (6) by $s_\lambda \longmapsto \sigma_\lambda$, using (19) and (20). This gives

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{m,\nu} q^m N^\nu_{\lambda\mu}(l,k)\sigma_\nu, \quad N^\nu_{\lambda\mu}(l,k) = \sum_{\varrho} \varepsilon(\varrho/\nu) N^\varrho_{\lambda\mu}. \tag{21}$$

Here $\lambda, \mu \subset l \times k$, the first sum is over $m \geq 0$ and $\nu \subset l \times k$ with

$$|\lambda| + |\mu| = |\nu| + mn,$$

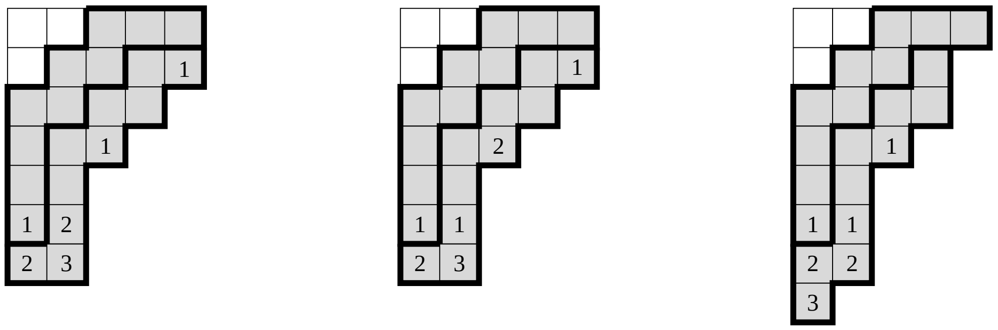
the second sum is over ϱ with $\varrho_1 \leq k$ obtained from ν by adding m n-rim hooks, and $N^\varrho_{\lambda\mu}$ are the *classical* Littlewood-Richardson coefficients. This gives a formula for the quantum Littlewood-Richardson numbers in terms of the classical numbers. From the identification of $\Lambda(l,k)$ with $QH^*(X)$ we deduce:

Corollary *For $\lambda, \mu, \nu \subset l \times k$, with $|\lambda| + |\mu| = |\nu| + mn$ for some $m \geq 0$, the number of rational curves of degree m that meet general translates of the Schubert varieties Ω_λ, Ω_μ, and $\Omega_{\nu^\vee}$ is equal to*

$$\sum \varepsilon(\varrho/\nu) N^\varrho_{\lambda\mu},$$

where the sum is over all ϱ with $\varrho_1 \leq k$ that can be obtained from ν by adding m n-rim hooks. In particular, $\sum \varepsilon(\varrho/\nu) N^\varrho_{\lambda\mu} \geq 0$. □

Example 1 $k = 5$, $l = 5$, $\lambda = (5,4,4,2,2)$, $\mu = (3,2,1)$, $\nu = (2,1)$; the only two shapes contributing to $N^\nu_{\lambda\mu}(5,5)$ are $\varrho_1 = (5,5,4,3,2,2,2)$ and $\varrho_2 = (5,4,4,3,2,2,2,1)$. These are shown below, together with the all the Littlewood-Richardson tableaux on ϱ_i/λ.



$\varepsilon = 1$ $\varepsilon = 1$ $\varepsilon = -1$

Hence $N_{\lambda\mu}^{\nu}(5,5) = 2 - 1 = 1$.

Example 2 This example exhibits the dependence of $N_{\lambda\mu}^{\nu}(l,k)$ on both l and k, as well as an instance where a quantum Littlewood-Richardson number (which is a nontrivial sum of signed classical ones) vanishes. Let $\lambda = (3,3,2,1)$, $\mu = (4,3,2,1)$, $\nu = (4,2,2,1)$. By removing a 10-rim hook from each of the partitions $\varrho_1 = (6,5,3,3,2)$, $\varrho_2 = (5,5,3,3,2,1)$, and $\varrho_3 = (4,4,3,3,2,1,1,1)$, one obtains the shape ν, and these are the only partitions with this property that appear in the product $s_\lambda \cdot s_\mu$. The classical Littlewood-Richardson coefficients are

$$N_{\lambda\mu}^{\varrho_1} = 6, \qquad N_{\lambda\mu}^{\varrho_2} = 8, \qquad N_{\lambda\mu}^{\varrho_3} = 2.$$

Our algorithm gives

$$N_{\lambda\mu}^{\nu}(4,6) = N_{\lambda\mu}^{\varrho_1} - N_{\lambda\mu}^{\varrho_2} + N_{\lambda\mu}^{\varrho_3} = 6 - 8 + 2 = 0,$$

$$N_{\lambda\mu}^{\nu}(5,5) = N_{\lambda\mu}^{\varrho_2} - N_{\lambda\mu}^{\varrho_3} = 8 - 2 = 6,$$

and

$$N_{\lambda\mu}^{\nu}(6,4) = N_{\lambda\mu}^{\varrho_3} = 2.$$

The quantum Pieri formula, for multiplying any σ_λ by a class σ_p, was given in [3]: for $\lambda \subset l \times k$, $1 \leq p \leq k$,

$$\sigma_p \cdot \sigma_\lambda = \sum \sigma_\mu + \sum q\sigma_\nu. \tag{22}$$

Here the first (classical) sum is over μ with

$$k \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \ldots \geq \mu_l \geq \lambda_l \geq 0,$$

and $|\mu| = |\lambda| + p$; the second sum is over ν with

$$\lambda_1 - 1 \geq \nu_1 \geq \lambda_2 - 1 \geq \nu_2 \geq \ldots \geq \lambda_l - 1 \geq \nu_l \geq 0,$$

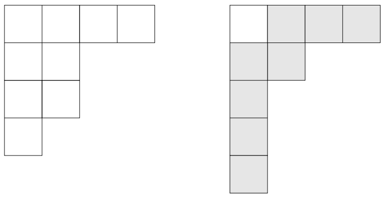
and $|\nu| = |\lambda| + p - n$. These ν are the partitions obtained from λ by removing $n - p$ boxes from its border rim, taking at least one box from each of the l rows.

The proof in [3] combined algebra with some quantum geometry. This Pieri formula follows easily from our Main Lemma. Indeed, one has the formula for multiplying Schur functions

$$s_p \cdot s_\lambda = \sum s_\mu,$$

the sum over μ obtained from λ by adding p boxes, no two in the same column. Terms with $\mu_1 > k$, or with $\mu_1 < k$ and $\mu_{l+1} > 0$ vanish by (A) of the Main Lemma. Those with $\mu_{l+1} = 0$ give the first sum in (22). Those with $\mu_{l+1} > 0$ and $\mu_1 = k$ have an n-rim hook that occupies all k columns; by (B) of the Main Lemma, these give the second sum in (22).

Example 3 $k = 4$, $l = 4$, $\lambda = (2, 2, 1, 1)$, $p = 3$:



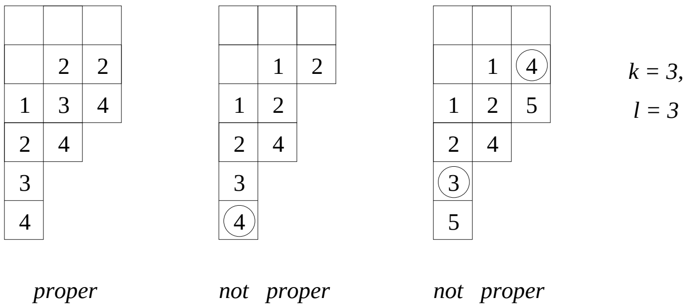
$$\sigma_3 \cdot \sigma_{(2,2,1,1)} = \sigma_{(4,2,2,1)} + q\sigma_1.$$

Note that the diagram of σ_1 is obtained from the shape $(2, 1, 1, 1)$ by removing 5 boxes.

3 Quantum Kostka numbers

For a partition $\nu \subset l \times k$, and a nonnegative integer m, we denote by $\nu[m]$ the partition obtained from ν by adding m n-rim hooks to ν, each starting in the first column and ending in the k^{th} column. Note that the height of each added rim is $l+1$, i.e., its lowest box in the first column is l rows below its highest box in the k^{th} column.

Let $\lambda \subset \varrho$ be partitions, with $\varrho_1 \leq k$. We call a (semistandard) tableau T on ϱ/λ *proper* if, for each entry i of T in the first column and row $l+p$, $p \geq 1$, the box in the k^{th} column and row p is either in λ, or it is in ϱ/λ and it has an entry of T which is at most equal to i.



For any sequence $\mu = (\mu_1, \ldots, \mu_r)$ of nonnegative integers, each at most k, and partitions $\lambda, \nu \subset l \times k$, with $|\lambda| + \sum \mu_i = |\nu| + mn$ for some $m \geq 0$, define the *quantum Kostka number* $K^{\nu}_{\lambda\mu}(l,k)$ to be the number of proper tableaux on $\nu[m]/\lambda$ with content μ. When $m = 0$, this is the classical Kostka number $K^{\nu}_{\lambda\mu}$ (the coefficient of s_ν in $s_{\mu_r} \cdot \ldots \cdot s_{\mu_1} \cdot s_\lambda$).

Proposition *For $\lambda \subset l \times k$ and $0 \leq \mu_1, \ldots, \mu_r \leq k$,*

$$\sigma_{\mu_r} \cdot \ldots \cdot \sigma_{\mu_1} \cdot \sigma_\lambda = \sum K^{\nu}_{\lambda\mu}(l,k) q^m \sigma_\nu,$$

the sum over $\nu \subset l \times k$ and $m \geq 0$, with $|\nu| + mn = |\lambda| + \sum \mu_i$.

Note in particular that $K^{\nu}_{\lambda\mu}(l,k)$ is independent of the ordering of the terms in $\mu = (\mu_1, \ldots, \mu_r)$.

The proof will follow from three lemmas. Fix λ and $\mu = (\mu_1, \ldots, \mu_r)$ as in the proposition. For any tableau T of content μ on a skew shape ϱ/λ, and $1 \leq i \leq r$, let T_i be the subtableau of T obtained by discarding all entries strictly larger than i; T_i is a tableau of content $(\mu_1, \ldots, \mu_i)$ on some shape $\varrho(i)/\lambda$. Set $\varrho(0) = \lambda$.

Lemma 1 *A tableau T of content μ on ϱ/λ is proper if and only if*

$$\widetilde{\varrho(i)}_1 - \widetilde{\varrho(i)}_k \leq l, \quad \textit{for} \quad 1 \leq i \leq r.$$

Proof: The condition of the definition is satisfied for an entry i in the first column of T if and only if the k^{th} column of $\varrho(i)$ is no more than l boxes shorter than its first column. $\square$

For any T that is not proper, let $U(T) = T_i$, where i is minimal not satisfying the condition of Lemma 1. We call U the *improper kernel* of T, and T a *descendent* of U if $U(T) = U$. Note that each $\varrho(i)$ has at most one more box in any column than $\varrho(i-1)$. It follows that if $U = U(T)$ has shape α/λ, then (since $\alpha = \varrho(i)$ and i is minimal)

$$\widetilde{\alpha}_1 - \widetilde{\alpha}_k = l + 1. \tag{23}$$

For any tableau T on a shape ϱ/λ, write σ_T for the element σ_ϱ in $\Lambda(l,k)$.

Lemma 2 *Let U be a tableau on α/λ with content $(\mu_1 \ldots, \mu_i)$ for some $1 \leq i \leq r$. Suppose $\widetilde{\alpha}_1 - \widetilde{\alpha}_k = l + 1$. Then*

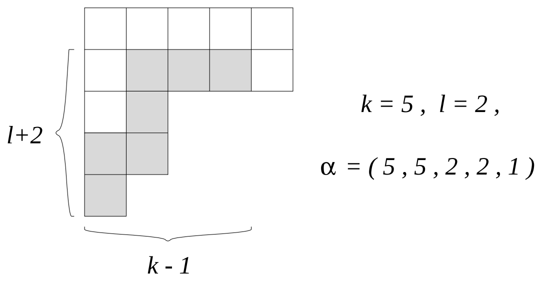
$$\sum \sigma_T = 0,$$

the sum over all T of content μ with $T_i = U$.

Proof: The sum $\sum \sigma_T$ in the statement is equal to

$$\sigma_{\mu_r} \cdot \sigma_{\mu_{r-1}} \cdot \ldots \cdot \sigma_{\mu_{i+1}} \cdot \sigma_\alpha$$

in $\Lambda(l,k)$. It therefore suffices to show that $\sigma_\alpha = 0$. From the assumption $\widetilde{\alpha}_1 - \widetilde{\alpha}_k = l + 1$ it follows that the n-rim starting at the bottom of the first column will end in the $(k-1)^{\text{st}}$ column, just to the left of the last box in the k^{th} column.



This is an illegal n-rim, so $\sigma_\alpha = 0$ by (A) of the Main Lemma. $\square$

Lemma 3 *Let ϱ be a partition with $\varrho_1 \leq k$ and $\widetilde{\varrho}_1 - \widetilde{\varrho}_k \leq l$. Then there is a unique partition $\nu \subset l \times k$ and $m \geq 0$ with $\varrho = \nu[m]$. Moreover, $\sigma_\varrho = q^m \sigma_\nu$ in $\Lambda(l,k)$.*

Proof: If $\widetilde{\varrho}_1 \leq l$, then $\nu = \varrho$ and $m = 0$. Otherwise, the condition $\widetilde{\varrho}_1 - \widetilde{\varrho}_k \leq l$ implies that ϱ contains an n-rim hook of width k and height $l+1$. Removing this rim hook leaves a shape β with $\widetilde{\beta}_1 - \widetilde{\beta}_k \leq l$, and $\sigma_\varrho = q\sigma_\beta$ by (B) of the Main Lemma. The proof concludes by induction on the number of boxes. $\square$

To prove the proposition, from the classical expansion of Schur functions (cf. [6], §2, or [12], §I.5), we have

$$\sigma_{\mu_r} \cdot \ldots \cdot \sigma_{\mu_1} \cdot \sigma_\lambda = \sum \sigma_T,$$

the sum over tableaux T of content μ on some shape $\varrho(T)/\lambda$. The shape $\varrho = \varrho(T)$ for any proper T satisfies the condition of Lemma 3 (by Lemma 1), so the sum of the σ_T for T proper gives the right side of the equation in the proposition. The T that are not proper are divided into equivalence classes indexed by their improper kernels $U = U(T)$. Each sum $\sum \sigma_T$ over T descended from U vanishes by Lemma 2, which concludes the proof of the proposition. $\square$

This proposition gives another algorithm for computing the quantum product $\sigma_\lambda \cdot \sigma_\mu$, for λ and μ partitions in $l \times k$: Expand

$$\sigma_\mu = \det(\sigma_{\mu_i+j-i}) = \sum_{\tau \in S_l} \operatorname{sign}(\tau) \prod_{i=1}^{l} \sigma_{\mu_i+\tau(i)-i}$$

and use the proposition to expand each $\prod \sigma_{\mu_i+\tau(i)-i} \cdot \sigma_\lambda$. This writes $\sigma_\lambda \cdot \sigma_\mu$ as an alternating sum of sums of σ_T's as T runs over proper tableaux of content $(\mu_1+\tau(1)-1, \ldots, \mu_l+\tau(l)-l)$ on shapes $\nu[m]/\lambda$. Note that among the sum with $\tau = id$ are all the tableaux whose word is a reverse lattice word. It is known from the classical Littlewood-Richardson rule that the other tableaux cancel in such a sum, but it is not clear how proper and nonproper tableaux behave under such a cancellation.

4 Duality

Although the functoriality of ordinary cohomology does not extend to quantum cohomology, isomorphic varieties do have isomorphic quantum cohomology rings. The canonical isomorphism of $Gr(l,n)$ with $Gr(k,n)$ gives an isomorphism of $QH^*(Gr(l,n))$ with $QH^*(Gr(k,n))$, which takes the class of a Schubert variety Ω_λ to the class of a Schubert variety $\Omega_{\widetilde{\lambda}}$, for $\lambda \subset l\times k$. Our basic algorithm from §2 is not at all invariant under the involution $\lambda \mapsto \widetilde{\lambda}$, however, so this isomorphism leads to some interesting combinatorial identities.

We begin with an algebraic proof of this isomorphism. Introducing a second set of variables, set

$$\Lambda(k,l) = \mathbf{Z}[q, \tau_1, \ldots, \tau_l]/(Y_{k+1}(\tau), \ldots Y_{n-1}(\tau), Y_n(\tau) + (-1)^l q).$$

Proposition 4.1 *The map $\sigma_i \mapsto Y_i(\tau)$, $1 \le i \le k$, determines an isomorphism $\Lambda(l,k) \xrightarrow{\sim} \Lambda(k,l)$, with inverse taking τ_i to $Y_i(\sigma)$, $1 \le i \le l$. For any $\lambda \subset l \times k$, this isomorphism takes $\sigma_\lambda = \det(\sigma_{\lambda_i+j-i})_{1\le i,j\le l}$ to $\tau_{\widetilde{\lambda}} = \det(\tau_{\widetilde{\lambda}_i+j-i})_{1\le i,j\le k}$.*

Proof: Consider the ring

$$\Lambda = \mathbf{Z}[q, \sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_l]/I\ ,$$

where I is generated by the elements

$$\sigma_1 - \tau_1,\ \sigma_2 - \sigma_1\tau_1 + \tau_2,\ \ldots,\ \sigma_k\tau_{l-1} - \sigma_{k-1}\tau_l,\ \sigma_k\tau_l - q.$$

These relations identify τ_i with $Y_i(\sigma)$ for $1 \le i \le l$, and then prescribe that $Y_i(\sigma) = 0$ for $l < i < n$, and that $Y_n(\sigma) = (-1)^{k-1}q$. This identifies Λ with $\Lambda(l,k)$. By symmetry, it also identifies Λ with $\Lambda(k,l)$, and the composite $\Lambda(l,k) \cong \Lambda \cong \Lambda(k,l)$ is the isomorphism of the proposition.

The fact that σ_λ and $\tau_{\widetilde{\lambda}}$ correspond under this isomorphism follows from the general identity [12], (2.9′), and the fact that all σ_i (resp. τ_i) occuring in the matrix for σ_λ (resp. $\tau_{\widetilde{\lambda}}$) have $i < n$. □

The following dual of the quantum Pieri formula follows from this proposition. We include a direct proof, for contrast with the proof of the Pieri formula (22) in Section 2.

Proposition 4.2 *Let $p \le l$ be a positive integer and let $\lambda \subset l \times k$ be a partition. The following holds in $\Lambda(l,k)$:*

$$\sigma_\lambda \cdot \sigma_{(1^p)} = \sum \sigma_\mu + q \sum \sigma_\nu, \tag{24}$$

the first sum over $\mu \subset l \times k$, obtained by adding p boxes to λ, with no two in the same row, and the second sum over ν obtained from λ by removing $n-p$ boxes in its border rim, with at least one from each of the k columns. Note that there are no such ν if $\lambda_1 < k$.

Equivalently, the partitions ν occuring in the second sum can be characterized by $|\nu| = |\lambda| + p - n$ and

$$\widetilde{\lambda}_1 - 1 \geq \widetilde{\nu}_1 \geq \widetilde{\lambda}_2 - 1 \geq \widetilde{\nu}_2 \geq \ldots \geq \widetilde{\lambda}_k - 1 \geq \widetilde{\nu}_k \geq 0.$$

Proof: The classical Pieri formula for Schur polynomials states that

$$s_\lambda \cdot s_{(1^p)} = \sum s_\pi, \tag{25}$$

sum over all π obtained by adding p boxes to λ, with no two in the same row. We need to show that the terms with $\pi_{l+1} > 0$ give the second sum in (24).

Let $\mathcal{P}$ denote the set of partitions π that occur in (25), with $\pi_1 \leq k$, $\pi_{l+1} > 0$, and the n-core of π contained in the $l \times k$ rectangle. Only partitions in $\mathcal{P}$ can contribute to the second sum in (24), by (A) of the Main Lemma. The formula (24) will follow therefore from the following two claims:

Claim 1. If ν is as in the proposition, then there exists a unique partition $\pi \in \mathcal{P}$ such that ν is obtained from π by removing an n-rim. Moreover, $\varepsilon(\pi/\nu) = 1$.

Claim 2. Let $\pi \in \mathcal{P}$ be any partition not arising in Claim 1. Then there exists exactly one other partition $\pi' \in \mathcal{P}$ with the same n-core ϱ as π. Moreover, $\varepsilon(\pi/\varrho) = -\varepsilon(\pi'/\varrho)$.

Proof of Claim 1: The partition π is obtained by adding p boxes to λ as follows: For each $i \in \{2, 3, \ldots, k\}$, add $\widetilde{\nu}_{i-1} - \widetilde{\lambda}_i + 1$ boxes to the i^{th} column of λ. The condition $\widetilde{\lambda}_1 - 1 \geq \widetilde{\nu}_1 \geq \widetilde{\lambda}_2 - 1 \geq \widetilde{\nu}_2 \geq \ldots \geq \widetilde{\lambda}_k - 1 \geq \widetilde{\nu}_k \geq 0$ ensures that no two of these new boxes are in the same row; the total number of boxes added according to this recipe is

$$\sum_{i=2}^{k} \widetilde{\nu}_{i-1} - \widetilde{\lambda}_i + 1 = |\nu| - \widetilde{\nu}_k - |\lambda| + \widetilde{\lambda}_1 + k - 1 = p - (l + 1 - \widetilde{\lambda}_1 + \widetilde{\nu}_k).$$

Now add $l + 1 - \widetilde{\lambda}_1 + \widetilde{\nu}_k$ boxes to the first column of λ to get a partition π.

Then π contains the n-rim hook consisting of these p added boxes together with the $n-p$ boxes in λ/ν; its width is k, so $\varepsilon(\pi/\nu)=1$. This is the only way a π in $\mathcal{P}$ can arise from ν by adding an n-rim hook.

Proof of Claim 2: Let π be a partition in $\mathcal{P}$. The n-rim of π starting at the bottom of the first column is an n-rim hook whose removal gives the n-core ϱ of π. This rim hook consists of some boxes in the rim of λ and some of the p added boxes. Let t be the maximum index such that this rim hook contains at least one box from the t^{th} column of λ.

If $t=k$, then π is one of the partitions considered in 1. We therefore assume that $t\neq k$. The rim hook must either (a) end with a box from λ in the t^{th} column of π, or (b) end with an added box in the $(t+1)^{\text{st}}$ column.

In case (a), π' is obtained from π by removing $\widetilde{\varrho}_t-\widetilde{\pi}_{t+1}+1$ boxes from the first column, and adding them to the $(t+1)^{\text{st}}$ column of π; note that π' satisfies (b), has the same n-core ϱ, and $\varepsilon(\pi/\varrho)=-\varepsilon(\pi'/\varrho)$.

Conversely, if π satisfies (b), the only π' with the same n-core as π is obtained from π by removing $\widetilde{\pi}_{t+1}-\widetilde{\varrho}_{t+1}$ boxes from the $(t+1)^{\text{st}}$ column and adding them to the first column. □

We turn next to a dual form of the proposition in Section 3; this is *not* obtained by applying the duality isomorphism of Proposition 4.1. Call a filling T of the boxes of a skew shape ϱ/λ a *conjugate tableau* if its entries are strictly increasing across rows and weakly increasing down columns. Call a conjugate tableau *proper* if for each entry i of T in the first column and row $l+p$, $p\geq 1$, the box in the k^{th} column and row p is either in λ, or it has an entry of T *strictly* smaller than i.

For a sequence $\mu=(\mu_1,\ldots,\mu_r)$ of nonnegative integers, and partitions $\lambda,\nu\subset l\times k$, with $|\lambda|+\sum\mu_i=|\nu|+mn$, $m\geq 0$, define the *conjugate quantum Kostka number* $\widetilde{K}^{\nu}_{\lambda\mu}(l,k)$ to be the number of proper conjugate tableaux on $\nu[m]/\lambda$ with content μ.

Proposition 4.3 *For $\lambda\subset l\times k$ and $0\leq\mu_1,\ldots,\mu_r\leq l$,*

$$\sigma_{(1^{\mu_r})}\cdot\ldots\cdot\sigma_{(1^{\mu_1})}\cdot\sigma_\lambda=\sum\widetilde{K}^{\nu}_{\lambda\mu}(l,k)q^m\sigma_\nu,$$

the sum over $\nu\subset l\times k$ and $m\geq 0$ with $|\nu|+mn=|\lambda|+\sum\mu_i$.

Proof: As in Section 3, for any conjugate tableau T on shape ϱ/λ with content μ, let T_i be the part of T of content $(\mu_1,\ldots,\mu_i)$, of shape $\varrho(i)/\lambda$.

This time T is proper if and only if, for $1 \le i \le r$, $\widetilde{\varrho(i)}_1 - \widetilde{\varrho(i)}_k \le l$, with the inequality strict if T has an entry i in the k^{th} column.

If T is not proper, let $U(T) = T_i$ for i minimal not satisfying the condition of the lemma; call T a descendent of $U(T)$. We consider conjugate tableaux U of content $(\mu_1, \ldots, \mu_i)$ which are not proper, but such that U_{i-1} (of content $(\mu_1, \ldots, \mu_{i-1})$) is proper and fixed. Consider the n-rim of the shape $\varrho(U)$ that starts in the first column. If this rim is illegal, then $\sigma_{\varrho(U)} = 0$. It follows as in Lemma 2 of §3 that $\sum \sigma_T = 0$, the sum over all T descended from U.

For each U whose n-rim is a rim hook we will construct another conjugate tableau U' of content $(\mu_1, \ldots, \mu_i)$, with $U'_{i-1} = U_{i-1}$, whose n-rim is also a rim hook. The results of removing the n-rim hooks from U' and U will be the same, but the rim hooks will end in adjacent columns, so $\sigma_{\varrho(U')} + \sigma_{\varrho(U)} = 0$. And we will have $(U')' = U$. It will follow that $\sum \sigma_T = 0$, the sum over all T descended from U or U'. The proof of the proposition concludes by an application of Lemma 3 of §3.

There are two cases, (a) and (b), which are interchanged by the involution $U \longleftrightarrow U'$. Let $V = U_{i-1}$, with shape α/λ (with $\alpha = \lambda$ if $i = 1$). Case (a) occurs when the n-rim in $\varrho(U)$ ends in a box of α, and (b) occurs when it ends in a box of $\varrho(U)/\alpha$. If U is as in case (a), the n-rim ends in a column t with $t < k$, since U is not proper; U' is obtained from U by taking $\widetilde{\varrho(U)}_t - \widetilde{\varrho(U)}_{t+1}$ i's from the end of the first column and putting them at the end of column $t+1$. (That this many i's can be removed from the first column follows from the fact that V is proper.) Conversely, if U is in case (b), and the n-rim ends in column $t+1$, the i's in the n-rim in column $t+1$ are removed and placed in the first column. $\square$

It follows from Proposition 4.1 that for $\lambda, \mu, \nu \subset l \times k$, with $|\lambda| + |\mu| = |\nu| + mn$, $m \ge 0$,

$$N_{\lambda\mu}^{\nu}(l,k) = N_{\widetilde{\lambda}\widetilde{\mu}}^{\widetilde{\nu}}(k,l). \tag{26}$$

Combining this with (the second part of) (21), and the duality $N_{\lambda\mu}^{\nu} = N_{\widetilde{\lambda}\widetilde{\mu}}^{\widetilde{\nu}}$ satisfied by the classical Littlewood-Richardson numbers, one gets the following nontrivial relation among these classical numbers:

$$\sum_{\varrho} \varepsilon(\varrho/\nu) N_{\lambda\mu}^{\varrho} = \sum_{\eta} \varepsilon(\eta/\widetilde{\nu}) N_{\lambda\mu}^{\widetilde{\eta}}, \tag{27}$$

where the sum on the left is over all ϱ with $\varrho_1 \le k$ that can be obtained from

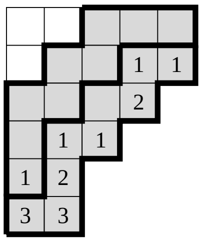
ν by adding m n-rim hooks, while the sum on the right is over all η with $\eta_1 \leq l$ that can be obtained from $\widetilde{\nu}$ by adding m n-rim hooks.

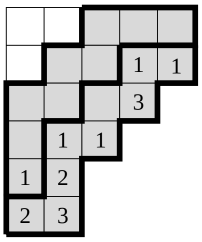
Similarly, from Proposition 4.3,

$$K^{\nu}_{\lambda\mu}(l,k) = \widetilde{K}^{\widetilde{\nu}}_{\widetilde{\lambda}\mu}(k,l), \tag{28}$$

for $\lambda, \nu \subset l \times k$, and $\mu = (\mu_1, \ldots, \mu_r)$, $0 \leq \mu_i \leq k$. The numbers on the right are the conjugate quantum Kostka numbers, calculated by regarding $\widetilde{\lambda}$ and $\widetilde{\nu}$ in the $k \times l$ rectangle. Equivalently, if $\nu[\widetilde{m}]$ is the partition obtained from ν by adding m n-rim hooks, each occupying rows 1 through l, then $\widetilde{K}^{\widetilde{\nu}}_{\widetilde{\lambda}\mu}(k,l)$ is the number of tableaux T of content μ on $\nu[\widetilde{m}]/\lambda$ such that for each entry i of T in the first row and column $k+p$, $p \geq 1$, the box in the l^{th} row and column p is either in λ, or it has an entry of T strictly smaller than i. We do not know a correspondance between the sets of tableaux that would explain equation (28). As in (27), equation (28) gives nontrivial relations among classical Kostka numbers.

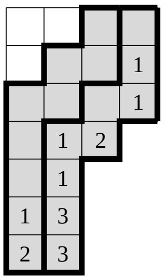
Example $k=5$, $l=4$, $\lambda=(5,3,3,1)$, $\mu=(5,2,2)$, $\nu=(2,1)$:

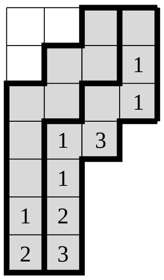




$$K^{\nu}_{\lambda\mu}(4,5) = 2.$$

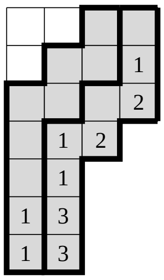
Note that there are in fact no nonproper tableaux of content μ on the shape $\nu[m]/\lambda$.





$$\widetilde{K}^{\tilde{\nu}}_{\lambda\mu}(5,4) = 2.$$

In contrast to the previous case, there are quite a few nonproper conjugate tableaux on $\tilde{\nu}[m]/\lambda$. One such tableau is



5 The Verlinde algebra and nonvanishing of products

C. Woodward has pointed out to us that the rim hook algorithm described in this paper provides an easy proof of the relation between the quantum cohomology ring of the Grassmannian $G(l,n)$ and the Verlinde algebra $R(\mathrm{U}(l)_k)$ of $\mathrm{U}(l)$ at level $k = n - l$. This Verlinde algebra is a free abelian group on the irreducible representations V_λ of $\mathrm{U}(l)$ with highest weight λ, where $k \geq \lambda_1 \geq \ldots \geq \lambda_l \geq 0$. The product, called the fusion product, writes V_λ times V_μ as a sum $\sum N(\lambda,\mu,\nu)V_\nu$, where $N(\lambda,\mu,\nu)$ is the dimension of the space of sections of a certain line bundle over a moduli space of $\mathrm{U}(l)$-bundles (see [2]). The result is that

$$QH^*(Gr(l,n))/(q-1) \cong R(\mathrm{U}(l)_k), \tag{29}$$

with σ_λ mapping to V_λ. This isomorphism was first observed by Gepner and Vafa, and was given a physical proof by Witten [15]. A rigorous mathematical proof was later found by Agnihotri [1].

Upon further searching in the literature, we discovered that algorithms similar to the one we describe here in the context of quantum cohomology (although in different formulations) have been given by Kac [11], Exercise 13.35 and Walton [14] in the case of fusion rings, and by Goodman and Wenzl in the context of representations of Hecke algebras at roots of unity [9]. A formulation involving removal of rim hooks which is closer (but not identical) to ours, again in the case of fusion rings, has been given by Cummins [5].

The following fact was noticed empirically by several people:

$$\text{The product } \sigma_\lambda \cdot \sigma_\mu \text{ is never zero in } \Lambda(l,k). \tag{30}$$

In terms of quantum cohomology, this means that for any two Schubert varieties Ω_λ and Ω_μ, there is some nonnegative integer d and some Schubert variety Ω_ν, with $|\nu| = |\lambda| + |\mu| - nd$, such that the quantum Littlewood-Richardson number $N^\nu_{\lambda\mu}(l,k)$ (a Gromov-Witten invariant of degree d) is positive.

C. Woodward first showed us a proof of this fact, based on the isomorphism (29) and showing that the fusion product of two irreducible representations is not zero. S. Agnihotri then gave a simple and direct proof, as follows. First note that if $\lambda^\vee = (k-\lambda_l, \ldots, k-\lambda_1)$ is the dual to λ, and $\rho = (k^l)$ is the class of a point, then $\sigma_\lambda \cdot \sigma_{\lambda^\vee} = \sigma_\rho + \sum_{d>0} q^d f_d$, with each f_d a nonnegative linear combination of classes σ_ν, with $|\nu| = kl - nd$. From this it suffices to prove that $\sigma_\rho \cdot \sigma_\rho \neq 0$. In fact it follows from the Main Lemma that $\sigma_\rho \cdot \sigma_\rho = q^l \sigma_{(k-l)^l}$ if $l \leq k$, and $\sigma_\rho \cdot \sigma_\rho = q^k \sigma_{k^{l-k}}$ if $k \leq l$.

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