DERIVED QUOT SCHEMES

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INTRODUCTION

(0.1) A typical moduli problem in geometry is to construct a "space" H parametrizing, up to isomorphism, objects of some given category $\mathcal{Z}$ (e.g., manifolds, vector bundles etc.). This can be seen as a kind of a non-Abelian cohomology problem and the construction usually consists of two steps of opposite nature, namely applying a left exact functor (A) followed by a right exact functor (B):

(A) One finds a space Z of "cocycles" whose points parametrize objects of $\mathcal{Z}$ equipped with some extra structure. Usually Z is given inside a much simpler space C of "cochains" by explicit equations, so forming Z is an inverse limit-type construction (hence left exact).

(B) One factorizes Z by the action of a group (or groupoid) B by identifying isomorphic objects and sets $H = Z/B$. This is a direct limit-type construction, hence right exact.

Part (B) leads to well known difficulties which in algebraic geometry are resolved by using the language of stacks. This can be seen as passing to the nonabelian left derived functor of (B). Indeed, an algebraic stack is a nonlinear analog of a complex of vector spaces situated in degrees $[-1, 0]$ and, for example, the tangent "space" to a stack at a point is a complex of this nature.

The step (A) may or may not be as clearly noticeable because points of C have no meaning from the point of view of the category $\mathcal{Z}$. It is also very important, nevertheless, because C is usually smooth while Z and hence H may be singular (even as a stack). For example, when $\mathcal{Z}$ consists of complex analytic vector bundles, we can take Z to consist of integrable $\bar{\partial}$-connections on a given smooth bundle. Then C consists of all $\bar{\partial}$-connections, integrable or not, which form an infinite-dimensional affine space, but do not, in general, define holomorphic bundles.

(0.2) The derived deformation theory (DDT) program, see [Kon] [Ka1] [Hi], is a program of research aimed at systematically resolving the difficulties related to singularities of the moduli spaces. It is convenient to formulate its most important premises as follows:

(a) One should take the right derived functor in the step (A) as well, landing in an appropriate "right derived category of schemes" whose objects (called dg-schemes) are nonlinear analogs of cochain complexes situated in degrees $[0, \infty)$ and whose tangent spaces are indeed complexes of this kind.

(b) The object RZ obtained in this way, should be manifestly smooth in an appropriate sense (so that the singular nature of Z is the result of truncation).

(c) The correct moduli "space" LRH is the result of applying to RZ the (stack-theoretic) left derived functor of (B). It should lie in a larger derived category of "dg-stacks" whose linear objects are cochain complexes situated in degrees $[-1, +\infty)$. The tangent space to LRH at a point corresponding to an object $X \in \mathcal{Z}$ is a complex of this kind, and its ith cohomology space is naturally identified to the $(i+1)$st cohomology space of the sheaf of infinitesimal automorphisms of X, thus generalizing the classical Kodaira-Spencer isomorphism to higher cohomology.

(d) All considerations in algebraic geometry which involve deformation to a generic almost complex structure can and should be replaced by systematically working with the derived moduli space LRH, its characteristic classes etc.

It is important not to confuse the putative dg-stacks of (c) with algebraic n-stacks as developed by Simpson [Si]: the latter serve as nonlinear analogs of cochain complexes situated in degrees $[-n, 0]$.

(0.3) In this paper we concentrate on taking the derived functor of the step (A) in the framework of algebraic geometry. Classically, almost all constructions of moduli spaces in this framework proceed via Hilbert schemes and their generalizations, *Quot* schemes, introduced by Grothendieck [Gr], see [Kol] [Vi] for detailed exposition. In many cases, the construction goes simply by quotienting an appropriate part of the *Quot* scheme by an action of an algebraic group, thus giving an algebraic stack. (Thus the scheme *Quot* plays the role of Z in (0.1)(A)).

The first step in constructing derived moduli spaces in algebraic geometry is, then, to construct the derived version of *Quot*. This is done in the present paper.

To recall the situation, let $\mathbb{K}$ be a field, let X be a projective scheme over $\mathbb{K}$ and $\mathcal{F}$ be a coherent sheaf on X. The scheme $Quot(\mathcal{F})$ can be viewed as parametrizing coherent subsheaves $\mathcal{K} \subset \mathcal{F}$: to every such $\mathcal{K}$ there corresponds a $\mathbb{K}$-point $[\mathcal{K}] \in Quot(\mathcal{F})$.

Assume that $\mathbb{K}$ has characteristic 0. For each Hilbert polynomial h we construct a smooth dg-manifold (see §2 for background) $RQuot_h(\mathcal{F})$ with the following properties:

(0.3.1) The degree 0 truncation $\pi_0 RQuot_h(\mathcal{F})$ is identified with $Quot_h(\mathcal{F})$.

(0.3.2) If $[\mathcal{K}]$ is a $\mathbb{K}$-point of $Quot_h(\mathcal{F})$ corresponding to a subsheaf $\mathcal{K} \subset \mathcal{F}$, then the tangent space $T^\bullet_{[\mathcal{K}]} RQuot_h(\mathcal{F})$ is a $\mathbb{Z}_+$-graded cochain complex whose cohomology is given by:

$$H^i T^\bullet_{[\mathcal{K}]} RQuot_h(\mathcal{F}) \simeq \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{K}, \mathcal{F}/\mathcal{K}).$$

Note that for the ordinary Quot scheme the tangent space is given by taking $i = 0$ in (0.3.2) (i.e., by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{F}/\mathcal{K})$). It is perfectly possible for the dimension of this Hom to jump in families (which causes singularities of *Quot*) but the Euler characteristic of Ext's is preserved under deformations (which explains the smoothness of *RQuot*).

(0.4) The derived *Quot* scheme we construct is suitable for construction of the derived moduli space of vector bundles. In the particular case of the Hilbert scheme, i.e., $\mathcal{F} = \mathcal{O}_X$, there is another natural derived version, $RHilb_h(X)$, which is suitable for construction of derived moduli spaces of algebraic varieties, (stable) maps etc. Its construction will be carried out in a sequel to this paper [CK]. To highlight the difference between $RHilb_h(X)$ and $RQuot_h(\mathcal{O}_X)$, take a $\mathbb{K}$-point of $Hilb_h(X)$ represented by a subscheme $Z \subset X$ with the sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$. Then, for a smooth Z and X it will be shown in [CK] that

$$H^i T^\bullet_{[Z]} RHilb_h(X) = H^i(Z, \mathcal{N}_{Z/X})$$

which is smaller than $\mathrm{Ext}^i_X(\mathcal{J}, \mathcal{O}_X/\mathcal{J}) = \mathrm{Ext}^{i+1}_X(\mathcal{O}_Z, \mathcal{O}_Z)$ which involves the cohomology of the higher exterior powers of the normal bundle.

(0.5) The paper is organized as follows. In Section 1 we give a background treatment of the *Quot* schemes. If we view *Quot* as an algebro-geometric instance of the space Z from (0.1)(A), then the role of the bigger space C is played by the ambient space of the Grassmannian embedding of *Quot* constructed by Grothendieck. We improve upon existing treatments by exhibiting an explicit system of equations of *Quot* in the product of Grassmannians (Theorem 1.4.1).

In Section 2 we make precise what we understand by the "right derived category of schemes" in which *RQuot* will lie. We develop the necessary formalism of smooth resolutions, homotopy fiber products etc.

In Section 3 we address a more algebraic problem: given an algebra A and a finite-dimensional A-module M, construct the derived version of the space (called the A-Grassmannian) parametrizing A-submodules in M of the given dimension. This construction will serve as a springboard for constructing the derived *Quot* scheme.

Finally, in Section 4 we give the construction of *RQuot*, using the approach of Section 3 and Theorem 1.4.1 which allows us to identify *Quot* with a version of the A-Grassmannian, but for a graded module over a graded algebra.

(0.6) The first published reference for the DDT program seems to be the paper [Kon] by M. Kontsevich, who gave an exposition of the ensuing "hidden smoothness

philosophy" in a lecture course in Berkeley in 1994. We are also aware of earlier unpublished suggestions of P. Deligne and V. Drinfeld containing very similar basic ideas. We gladly acknowledge our intellectual debt to our predecessors. We are also grateful to participants of the deformation theory seminar at Northwestern, where this work originated and was reported. Both authors were partially supported by NSF.

1. Grothendieck's Quot Scheme

(1.1) Elementary properties. We recall briefly the definition and main properties of Grothendieck's *Quot* scheme ([Gr], see also [Kol][Vi] for detailed treatments). Let $\mathbb{K}$ be a field and X be a projective scheme over $\mathbb{K}$, with a chosen very ample invertible sheaf $\mathcal{O}_X(1)$. For any coherent sheaf $\mathcal{G}$ on X denote as usual $\mathcal{G}(n) := \mathcal{G} \otimes \mathcal{O}_X(n)$. The *Hilbert polynomial* $h^{\mathcal{G}}$ of $\mathcal{G}$ is the polynomial in $\mathbb{Q}[t]$ defined by

$$h^{\mathcal{G}}(n) = \chi(\mathcal{G}(n)).$$

By Serre's vanishing theorem $h^{\mathcal{G}}(n) = \dim H^0(X, \mathcal{G}(n))$ for $n >> 0$.

Now fix a coherent sheaf $\mathcal{F}$ on X, a polynomial $h' \in \mathbb{Q}[t]$, and set $h := h^{\mathcal{F}} - h'$. Informally the *Quot* scheme can be thought of as a "Grassmannian of subsheaves in $\mathcal{F}$"; its closed points are in 1-1 correspondence with

$$Sub_h(\mathcal{F}) := \{\mathcal{K} \subset \mathcal{F} \mid h^{\mathcal{K}} = h\},$$

or, equivalently, with

$$Quot_{h'}(\mathcal{F}) := \{\mathcal{F} \twoheadrightarrow \mathcal{G} \mid h^{\mathcal{G}} = h'\}/Aut(\mathcal{G}).$$

The scheme structure reflects how quotients of $\mathcal{F}$ vary in families. More precisely, for any scheme S, let π_X denote the canonical projection $X \times S \longrightarrow X$. Grothendieck's theorem then states:

(1.1.1) Theorem. *There exists a projective scheme $Sub_h(\mathcal{F})$ (or $Quot_{h'}(\mathcal{F})$) such that for any scheme S we have*

$$Hom(S, Sub_h(\mathcal{F})) = \left\{ \mathcal{K} \subset \pi_X^*\mathcal{F} \;\middle|\; \begin{array}{c} \pi_X^*\mathcal{F}/\mathcal{K} \textit{ is flat over } S \textit{ and has} \\ \textit{relative Hilbert polynomial } h' \end{array} \right\}.$$

Thus, in particular, we have the *universal exact sequence* on $Sub_h(\mathcal{F}) \times X$, with $\mathcal{S}$ corresponding to the identity map $Sub_h(\mathcal{F}) \to Sub_h(\mathcal{F})$;

$$0 \longrightarrow \mathcal{S} \longrightarrow \pi_X^*\mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0. \tag{1.1.2}$$

The following statement is obtained easily by taking $S = \operatorname{Spec}(\mathbb{K}[x]/x^2)$ in Theorem 1.1.1, see [Gr, Cor. 5.3].

(1.1.3) Proposition. *Let $[\mathcal{K}]$ be the $\mathbb{K}$-point in $Sub_h(\mathcal{F})$ determined by a subsheaf $\mathcal{K} \subset \mathcal{F}$ with $h^{\mathcal{K}} = h$. Then the tangent space to $Sub_h(\mathcal{F})$ at $[\mathcal{K}]$ is*

$$T_{[\mathcal{K}]}Sub_h(\mathcal{F}) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{F}/\mathcal{K}).$$

(1.2) The Grassmannian embedding. Let W be a finite-dimensional vector space. By $G(k,W)$ we denote the Grassmannian of k-dimensional linear subspaces in W. Thus, to every such subspace $V \subset W$ there corresponds a point $[V] \in G(k,W)$. We denote by $\widetilde{V}$ the tautological vector bundle on $G(k,W)$ whose fiber over $[V]$ is V. It is well known that $T_{[V]}G(k,W) \simeq \mathrm{Hom}(V, W/V)$.

Let $X, \mathcal{O}(1)$ be as before. Set $A := \bigoplus_{i\geq 0} H^0(X, \mathcal{O}_X(i))$. This is a finitely generated graded commutative algebra. For a coherent sheaf $\mathcal{G}$ on X let $\mathrm{Mod}(\mathcal{G}) = \bigoplus_i H^0(X, \mathcal{G}(i))$ be the corresponding graded A-module. Similarly, for a finitely generated graded A-module M we denote by $\mathrm{Sh}(M)$ the coherent sheaf on X corresponding to M by localization.

If M is a graded A-module, we denote $M_{\geq p}$ the submodule consisting of elements of degree at least p. Similarly, for $p \leq q$ we set $M_{[p,q]} = M_{\geq p}/M_{\geq q}$ to be the truncation of M in degrees $[p,q]$.

Given finitely generated graded A-modules M, N we define

$$\mathrm{Hom}_{\mathcal{S}}(M,N) = \varinjlim_{p} \mathrm{Hom}^0(M_{\geq p}, N_{\geq p}), \tag{1.2.1}$$

where Hom^0 is the set of A-homomorphisms of degree 0.

Recall the classical theorem of Serre [Se, §59].

(1.2.2) Theorem. *The category $Coh(X)$ of coherent sheaves on X is equivalent to the category $\mathcal{S}$ whose objects are finitely generated graded A-modules and morphisms are given by (1.2.1). More precisely, if M, N are objects of $\mathcal{S}$, then*

$$\mathrm{Hom}_{\mathcal{S}}(M,N) = \mathrm{Hom}_{\mathcal{O}_X}(Sh(M), Sh(N)).$$

Further, the limit in (1.2.1) is achieved for some $p = p(M,N)$.

Part (a) of the following theorem is also due to Serre [Se, §66].

(1.2.3) Theorem. *(a) For any coherent sheaf $\mathcal{G}$ on X there exists an integer $p = p(\mathcal{G})$ such that $H^j(X, \mathcal{G}(r)) = 0$ for all $j \geq 0$ and all $r \geq p$, and the multiplication map*

$$H^0(X, \mathcal{O}_X(i)) \otimes H^0(X, \mathcal{G}(r)) \longrightarrow H^0(X, \mathcal{G}(r+i))$$

is surjective for all $i \geq 0$ and all $r \geq p$.

(b) The number p in part (a) can be chosen uniformly with the above properties for all subsheaves $\mathcal{K}$ of a fixed coherent sheaf $\mathcal{F}$ on X with fixed Hilbert polynomial $h^{\mathcal{K}} = h$, and for all respective quotients $\mathcal{F}/\mathcal{K}$.

Part (b) is proved in [Mu, Lecture 14] or [Vi, Thm. 1.33]. More precisely, the discussion of [Vi] is, strictly speaking, carried out only for the case $\mathcal{F} = \mathcal{O}_X^n$. This, however, implies the case $\mathcal{F} = \mathcal{O}_X(i)^n$ for any i and n and then the case of an arbitrary $\mathcal{F}$ follows from this by taking a surjection $\mathcal{O}_X(i)^n \to \mathcal{F}$.

In terms of the associated module $N = \mathrm{Mod}(\mathcal{G})$, part (a) means that $N_{\geq p}$ is generated by N_p and $\dim N_r = h^{\mathcal{G}}(r)$ for $r \geq p$.

Fix now a coherent sheaf $\mathcal{F}$ and a polynomial h and pick p such as in (1.2.3)(b) which is large enough so that the statements of (a) hold for $\mathcal{F}$ as well. Consider the universal exact sequence (1.1.2). For $r \geq p$, twisting by $\pi_X^* \mathcal{O}_X(r)$ and pushing forward to $Sub_h(\mathcal{F})$ produces an exact sequence of *vector bundles*

$$0 \longrightarrow (\pi_{Sub})_* \mathcal{S}(r) \longrightarrow M_r \otimes \mathcal{O}_{Sub} \longrightarrow (\pi_{Sub})_* \mathcal{Q}(r) \longrightarrow 0,$$

with $\mathrm{rank}(\pi_{Sub})_* \mathcal{S}(r) = h(r)$, which in turn determines a map

$$\alpha_r : Sub_h(\mathcal{F}) \longrightarrow G(h(r), M_r), \tag{1.2.4}$$

Now Grothendieck's Grassmannian embedding is as follows.

(1.2.5) Theorem. *For $r \gg 0$ the map α_r identifies $Sub_h(\mathcal{F})$ with a closed subscheme of the Grassmannian $G(h(r), M_r)$.*

(1.3) The A-Grassmannian. We now discuss a more elementary construction which can be seen as a finite-dimensional analog of the *Quot* scheme.

Let A be an associative algebra over $\mathbb{K}$ (possibly without unit) and M be a finite-dimensional left A-module. The A-Grassmannian is the closed subscheme $G_A(k, M) \subset G(k, M)$ formed by those k-dimensional subspaces which are left A-submodules. It can be defined as the (scheme-theoretical) zero locus of the canonical section

$$s \in \Gamma(G(k, M), \mathcal{H}om(A \otimes_{\mathbb{K}} \widetilde{V}, M/\widetilde{V})) \tag{1.3.1}$$

whose value over a point $[V]$ is the composition of the A-action $A \otimes V \to M$ with the projection $M \to M/V$. It follows that if V is a submodule, then

$$T_{[V]} G_A(k, M) = \mathrm{Hom}_A(V, M/V) \subset \mathrm{Hom}_{\mathbb{K}}(V, M/V) = T_{[V]} G(k, M). \tag{1.3.2}$$

This is similar to (1.1.3).

Next, suppose that $M = \bigoplus_i M_i$ is a finite-dimensional $\mathbb{Z}$-graded vector space, i.e., each M_i is finite-dimensional and $M_i = 0$ for almost all i. Let $k = (k_i)$ be a sequence of nonnegative integers. We denote $G(k, M) = \prod G(k_i, M_i)$; in other words, this is the variety of graded subspaces $V = \bigoplus V_i \subset M$ such that $\dim(V_i) = k_i$. As before, we denote by $[V]$ the point of $G(k, M)$ represented by a graded subspace V, and denote by $\widetilde{V} = \bigoplus \widetilde{V}_i$ the tautological graded vector bundle over $G(k, M)$.

Let now $A = \bigoplus_i A_i$ be a $\mathbb{Z}$-graded associative algebra and $M = \bigoplus_i M_i$ be a finite-dimensional graded left A-module, We have then the graded A-Grassmannian

$G_A(k,M) \subset G(k,M)$ parametrizing graded A-submodules $V \subset M$. It can be defined as the common zero locus of the natural sections s_{ij} of the bundles $\mathcal{H}om(A_i \otimes \widetilde{V}_j, M_{i+j}/\widetilde{V}_{i+j})$. For a submodule V we have

$$T_{[V]} G_A(k,M) = \mathrm{Hom}_A^0(V, M/V), \tag{1.3.3}$$

where Hom_A^0 means the set of homomorphisms of degree 0.

(1.4) ***Quot*** **as an** ***A*****-Grassmannian.** We specialize the considerations of (1.3) to the case

$$A = \bigoplus H^0(X, \mathcal{O}_X(i)), \quad M = \bigoplus_i H^0(X, \mathcal{F}(i)), \quad \mathcal{F} \in \mathrm{Coh}(X)$$

from (1.2). Let $p > 0$ be chosen as in (1.2). For $p \le q$ the morphism

$$\alpha_{[p,q]} = \prod_{r=p}^{q} \alpha_r : Sub_h(\mathcal{F}) \to \prod_{r=p}^{q} G(h(r), M_r) = G(h, M_{[p,q]})$$

takes values, by construction, in the A-Grassmannian $G_A(h, M_{[p,q]})$. The following result extends Theorem 1.2.5 by providing explicit relations for the Grassmannian embedding of *Quot*. It seems not to be found in the literature. A related statement (which does not take into account the nilpotents in the structure sheaves of the schemes involved and assumes $\mathcal{F} = \mathcal{O}_X$), is due to Gotzmann [Go, Bemerkung 3.3]. In his situation it is enough to take $q = p+1$.

(1.4.1) Theorem. *For $0 \ll p \ll q$ the morphism $\alpha_{[p,q]} : Sub_h(\mathcal{F}) \to G_A(h, M_{[p,q]})$ is an isomorphism.*

First notice that we may assume that $\mathbb{K}$ is algebraically closed. Before giving the proof of the theorem, we need some preparations. To unburden the notation, for $q \ge p$ set

$$G_q := G_A(h, M_{[p,q]}).$$

In particular $G_p = G(h(p), M_p)$, since $M_{[p,p]} = M_p$ has trivial A-module structure.

For $r \ge s \ge p$, let $\varphi_{rs} : G_r \longrightarrow G_s$ be the canonical projection. We have then an *inverse system* of schemes

$$G_p \longleftarrow G_{p+1} \longleftarrow \cdots \longleftarrow G_r \longleftarrow \cdots . \tag{1.4.2}$$

(1.4.3) Lemma. *The projective limit of the diagram (1.4.2) in the category of schemes exists and is identified with $Sub_h(\mathcal{F})$.*

Proof. We have to show that for any scheme S, a compatible system of maps $S \longrightarrow G_r$, $r \geq p$ gives rise to a map $S \longrightarrow Sub_h(\mathcal{F})$. But such a system gives rise to a family (parametrized by S) of graded A-submodules of $M_{\geq p}$ with Hilbert polynomial (in fact, even the Hilbert function) equal to h, i.e., to a graded $A \otimes \mathcal{O}_S$-submodule $\mathcal{V} \subset M \otimes \mathcal{O}_S$ such that each $\mathcal{V}_i$ is a projective $\mathcal{O}_S$-module (i.e., a vector bundle on S) of rank $h(i)$. It follows that each $(M_i \otimes \mathcal{O}_S)/\mathcal{V}_i$ is projective, as an $\mathcal{O}_S$-module. Thus the graded $\mathcal{O}_S$-module $(M \otimes \mathcal{O}_S)/\mathcal{V}$ is flat (because it is the union of projective $\mathcal{O}_S$-submodules $(M \otimes \mathcal{O}_S)_{\leq p}/\mathcal{V}_{\leq p}$). By Serre's theorem $\mathcal{V}$ gives a subsheaf $\mathcal{K} \subset \pi^*\mathcal{F}$ and the quotient sheaf $\pi^*\mathcal{F}/\mathcal{K}$, corresponding to the $\mathcal{O}_S$-flat graded $A \otimes \mathcal{O}_S$-module $(M \otimes \mathcal{O}_S)/\mathcal{V}$, is also flat over $\mathcal{O}_S$. Hence we get the required map $S \longrightarrow Sub_h(\mathcal{F})$.

Recall that any morphism $f : Y \to Z$ of projective schemes has a well-defined image which is a closed subscheme $\mathrm{Im}(f) \subset Z$ satisfying the usual categorical universal property.

With this understanding, for any $r \geq p$, we consider the subscheme $\widetilde{G}_r$ of G_r defined by

$$\widetilde{G}_r := \bigcap_{r' \geq r} \mathrm{Im}\{\varphi_{r'r} : G_{r'} \longrightarrow G_r\}. \tag{1.4.4}$$

Because of the Noetherian property, the intersections in (1.4.4) in fact stabilize.

(1.4.5) Lemma. *Together with the restrictions of the natural projections, the subschemes $\widetilde{G}_r$ form an inverse system of surjective maps with the same projective limit $Sub_h(\mathcal{F})$ as the system (1.4.2).*

Proof. This is a purely formal argument. We consider (G_r) and $(\widetilde{G}_r)$ as pro-objects in the category of schemes (see [GV], §8) and will show that they are isomorphic in the category of pro-objects. This will imply that $\varprojlim \widetilde{G}_r$ exists and is isomorphic to $\varprojlim G_r$. First, the componentwise morphism of inverse systems $(u_r : \widetilde{G}_r \to G_r)$ gives a morphism of pro-objects, which we denote u_*. Next, stabilization of the images implies that for every r there is a $q = q(r)$ and a morphism $v_r : G_{q(r)} \to \widetilde{G}_r$. These constitute a morphism of pro-objects $v_* : (G_r) \to (\widetilde{G}_r)$, which one checks is inverse to u_*.

(1.4.6) Corollary. *The projective system $(\widetilde{G}_r)$ is constant. In particular, for any $r \geq p$ the natural projection $\widetilde{\varphi}_r : Sub_h(\mathcal{F}) \to \widetilde{G}_r$ is an isomorphism.*

Proof. This follows from the previous two lemmas and Grothendieck's theorem 1.2.5 on the Grassmannian embedding which can be formulated by saying that $\widetilde{\varphi}_p : Sub_h(\mathcal{F}) \to \widetilde{G}_p$ is an isomorphism.

Proof of Theorem (1.4.1). It follows from Lemmas 1.4.3 and 1.4.5 and Corollary 1.4.6 that we have a commutative diagram

$$\begin{array}{ccccccc} & & & & & & Sub_h(\mathcal{F}) \\ & & & & & & \downarrow \\ \widetilde{G}_p & \overset{\sim}{\longleftarrow} & \widetilde{G}_{p+1} & \overset{\sim}{\longleftarrow} & \cdots & \overset{\sim}{\longleftarrow} & \varprojlim \widetilde{G}_r \\ \bigcap & & \bigcap & & & & || \\ G_p & \longleftarrow & G_{p+1} & \longleftarrow & \cdots & \longleftarrow & \varprojlim G_r \end{array}$$

such that for every $r \geq p$ the induced map $Sub_h(\mathcal{F}) \to \widetilde{G}_r$ is an isomorphism and $\alpha_{[p,r]}$ factors as $Sub_h(\mathcal{F}) \to \widetilde{G}_r \hookrightarrow G_r$. Let q be such that for any $r \geq q$

$$\mathrm{Im}\{\varphi_{rp} : G_r \longrightarrow G_p\} = \widetilde{G}_p(\cong Sub_h(\mathcal{F})).$$

For an A-submodule $V = \bigoplus_{i=p}^r V_i$ of $M_{[p,r]}$ let $\vec{V}$ be the A-submodule of $M_{\geq p}$ generated by V. Let now $W = W_p \in \widetilde{G}_p$ be a subspace of M_p. Since $\widetilde{G}_p \cong Sub_h(\mathcal{F})$ we have $W = H^0(X, \mathcal{K}(p))$, for some $\mathcal{K} \subset \mathcal{F}$ with $h^{\mathcal{K}} = h$. By our choice of p, it follows that $\vec{W} = \bigoplus_{i \geq p} H^0(X, \mathcal{K}(i))$.

(1.4.7) Lemma. *For $r \geq q$ the set-theoretic fibre $\varphi_{rp}^{-1}(W) \subset G_r$ consists of the unique $\mathbb{K}$-point $(\vec{W})_{[p,r]} := \vec{W} \cap M_{[p,r]}$.*

Granting this for a moment, it follows that for $r \geq q$ the map φ_{rp} is a bijection on $\mathbb{K}$-points onto $\widetilde{G}_p$, and therefore $\alpha_{[p,r]}$ is also a bijection on $\mathbb{K}$-points onto $G_r = G_A(h, M_{[p,r]})$. This gives Theorem 1.4.1 at the level of sets. To prove it in general, consider the tautological family of A-submodules of $M_{[p,r]}$ over $G_A(h, M_{[p,r]})$ obtained by restricting the tautological vector subbundle over $G(h, M_{[p,r]})$. This determines (by pull-back to $G_A(h, M_{[p,r]}) \times X$ and application of the functor " $\vec{*}$ ") a family of A-submodules of $M_{\geq p}$ with Hilbert polynomial equal to h. The same argument as in the proof of Lemma 1.4.3 gives then a map $\beta_{[p,r]} : G_A(h, M_{[p,r]}) \longrightarrow Sub_h(\mathcal{F})$ which is easily seen to be an inverse for $\alpha_{[p,r]}$.

Proof of Lemma 1.4.7. Let $V \in G_A(h, M_{[p,r]})$ be such that $V \cap M_p = W$. Then $\vec{W} \subset \vec{V} \subset M_{\geq p}$. Since V is an A-submodule, it follows that $(\vec{W})_{[p,r]} \subset V$. But for each $p \leq i \leq r$ the dimension over $\mathbb{K}$ of the graded components of degree i of this last two modules is the same, therefore $(\vec{W})_{[p,r]} = V$.

This concludes the proof of Theorem 1.4.1.

2. The right derived category of schemes.

In this section we develop the minimal necessary background suitable for taking right derived functors on the category of schemes (which correspond to left derived functors on the category of commutative algebras).

(2.1) Dg-vector spaces, algebras and modules. From now on we assume[1] that the base field $\mathbb{K}$ has characteristic 0. By a complex (or dg-vector space) we always mean a cochain complex, i.e., a graded vector space C with a differential of degree $+1$. The grading of complexes will be always indicated in the superscript, to distinguish it from other types of grading which may be eventually present (such as in (1.2) above). If C is a complex and $a \in C^i$, we write $\bar{a} = i$. We also write $H(C)$ for the graded space of cohomology of C and $C_\#$ for the graded vector space obtained from C by forgetting the differential. A morphism $f : C \to D$ of complexes is called a quasiisomorphism if $H(f) : H(C) \to H(D)$ is an isomorphism.

Complexes form a symmetric monoidal category $dgVect$ with respect to the usual tensor product and the symmetry operator given by the Koszul sign rule: $a \otimes b \mapsto (-1)^{\bar{a}\bar{b}} b \otimes a$. By an associative resp. commutative dg-algebra we mean an associative, resp. commutative algebra in $dgVect$. By a graded algebra we mean a dg-algebra with zero differential. Thus for every dg-algebra A we have graded algebras $H(A), A_\#$. Note that a graded commutative algebra in this sense satisfies $ab = (-1)^{\bar{a}\bar{b}} ba$.

Similar conventions and terminology will be used for dg-modules over a dg-algebra A (left or right, if A is not commutative).

In this paper we will always consider (unless otherwise specified), only dg-algebras A which are $\mathbb{Z}_-$-graded, i.e., have $A^i = 0$ for $i > 0$.

The following remark, though obvious, is crucial for gluing commutative dg-algebras into more global objects.

(2.1.1) Proposition. *If A is a $\mathbb{Z}_-$-graded associative dg-algebra and M is a left dg-module over A, then each $d : M^i \to M^{i+1}$ is A^0-linear.*

Let A be an associative dg-algebra. A left dg-module M over A is called quasifree, if $M_\#$ is free over $A_\#$, so as a graded module, $M = A \otimes_{\mathbb{K}} E^\bullet$, where $E^\bullet$ is some graded vector space of generators.

Let M, N be left dg-modules over A. Morphisms of dg-modules $M \to N$ are degree 0 cocycles in the cochain complex $\mathrm{Hom}^\bullet_A(M, N)$ which consists of all A-linear morphisms and whose differential is given by the commutation with the differentials in M, N. Two morphisms $f, g : M \to N$ are called homotopic, if they

[1] The reasons for the characteristic 0 assumptions are the standard ones in the theory of dg-algebras [Le] [Q2]. For example, the construction of M-homotopies in (3.6) requires taking anti-derivatives of polynomials with coefficients in $\mathbb{K}$.

are cohomologous as cocycles, i.e., if there exists a morphism $s : M \to N[-1]$ of $A_\#$-modules such that $d_N s + s d_M = f - g$.

(2.1.2) Proposition. *Let M, N be two $\mathbb{Z}_-$-graded left dg-modules over A, and suppose that M is quasifree and N is acyclic in degrees < 0. Let $f, g : M \to N$ be dg-morphisms which induce the same morphism $H^0(M) \to H^0(N)$. Then f is homotopic to g.*

Proof: This is a standard inductive construction and is left to the reader. In fact, in Proposition 3.6.4 we give a less trivial, nonlinear version of this construction and spell out the proof in full detail. The reader can easily adapt that proof to the present linear situation.

(2.1.3) Corollary. *If M is a quasifree dg-module over A which is bounded from above and which is exact with respect to its differential, then M is contractible, i.e., its identity map is homotopic to 0.*

Proof: By shifting the degree we can assume that M is $\mathbb{Z}_-$-graded. Then apply (2.1.2) to $M = N$ and to $f = \mathrm{Id}_M$, $g = 0$.

(2.1.4) Corollary. *A quasiisomorphism $f : M \to N$ of quasifree, bounded from above dg-modules over A, is a homotopy equivalence.*

Proof: Consider the dg-module $\mathrm{Cone}(f)$. It is acyclic, quasifree and bounded from above, so contractible by (2.1.3). This implies our statement.

(2.1.5) Corollary. *In the situation of Corollary 2.1.4, the dual morphism*

$$f^* : \mathrm{Hom}_A(N, A) \to \mathrm{Hom}_A(M, A)$$

is a quasiisomorphism.

Proof: As f is a homotopy equivalence, so is f^*, because homotopies are inherited under functorial constructions on modules such as $\mathrm{Hom}_A(-, A)$.

(2.2) Dg-schemes.

(2.2.1) Definition. *(a) By a dg-scheme we mean a pair $X = (X^0, \mathcal{O}_X^\bullet)$, where X^0 is an ordinary scheme and $\mathcal{O}_X^\bullet$ is a sheaf of ($\mathbb{Z}_-$-graded) commutative dg-algebras on X^0 such that $\mathcal{O}_X^0 = \mathcal{O}_{X^0}$ and each $\mathcal{O}_X^i$ is quasicoherent over $\mathcal{O}_X^0$.*

(b) A morphism $f : X \to Y$ of dg-schemes is just a morphism of (dg-) ringed spaces, i.e., a morphism $f_0 : X^0 \to Y^0$ of schemes together with a morphism of sheaves of dg-algebras $f_0^* \mathcal{O}_Y^\bullet \to \mathcal{O}_X^\bullet$. The category of dg-schemes will be denoted by $dgSch$.

By a graded scheme we mean a dg-scheme X in which $\mathcal{O}_X^\bullet$ has trivial differential. Any ordinary scheme will be considered as a dg-scheme with trivial grading and differential.

By (2.1.1), for a dg-scheme X each $d : \mathcal{O}_X^i \to \mathcal{O}_X^{i+1}$ is $\mathcal{O}_{X^0}$-linear and hence $\underline{H}^i(\mathcal{O}_X^\bullet)$ are quasicoherent sheaves on X^0. We define the "degree 0 truncation" of X to be the ordinary scheme

$$\pi_0(X) = \mathrm{Spec}\underline{H}^0(\mathcal{O}_X^\bullet) \subset X^0. \tag{2.2.2}$$

The notation is chosen to suggest analogy with homotopy groups in topology. Note that for any ordinary scheme S we have

$$\mathrm{Hom}_{dgSch}(S, X) = \mathrm{Hom}_{Sch}(S, \pi_0(X)). \tag{2.2.3}$$

Note also that each $\underline{H}^i(\mathcal{O}_X^\bullet)$ can be regarded as a quasicoherent sheaf on $\pi_0(X)$. We have then two graded schemes naturally associated to X:

$$X_\# = (X^0, \mathcal{O}_{X,\#}^\bullet), \quad X_h = (\pi_0(X), \underline{H}^\bullet(\mathcal{O}_X^\bullet)). \tag{2.2.4}$$

A morphism $f : X \to Y$ of dg-schemes will be called a quasiisomorphism if the induced morphism of graded schemes $f_h : X_h \to Y_h$ is an isomorphism. We denote by $\mathcal{D}Sch$ the category obtained from $dgSch$ by inverting all quasiisomorphisms and call it the (right) derived category of schemes. It is suitable for taking right derived functors on schemes (which correspond to left derived functors at the level of commutative algebras).

(2.2.5) Examples. (a) If $A^\bullet$ is a commutative $\mathbb{Z}_-$-graded dg-algebra, we have a dg-scheme $X = \mathrm{Spec}(A^\bullet)$ defined as follows. The scheme X^0 is $\mathrm{Spec}(A^0)$, and the sheaf $\mathcal{O}_X^i$ is the quasicoherent sheaf on $\mathrm{Spec}(A^0)$ associated to the A^0-module A^i. A dg-scheme X having the form $\mathrm{Spec}(A^\bullet)$ will be called affine. We will also write $A^\bullet = \mathbb{K}[X]$ and call A the coordinate (dg-) algebra of X.

(b) In particular, if $E^\bullet$ is a $\mathbb{Z}_+$-graded complex of finite-dimensional vector spaces, we have the dg-scheme $|E| = \mathrm{Spec}\ S(E^*)$, which is "the linear dg-space $E^\bullet$ considered as a scheme". We will extend this notation as follows. Let $F^\bullet$ be a $\mathbb{Z}_-$-graded complex of possibly infinite-dimensional vector spaces. Then we write $|F^*| = \mathrm{Spec}\ S(F)$. We will use this notation for ungraded vector spaces as well.

(2.2.6) Definition. *A dg-scheme X is said to be of finite type, if X^0 is a scheme of finite type, and each $\mathcal{O}_X^i$ is coherent, as a sheaf on X^0. We say that X is (quasi)projective if it is of finite type and X^0 is (quasi)projective in the usual sense.*

(2.3) Dg-sheaves. We now globalize the usual theory of dg-modules over a dg-algebra [HMS] [Mc].

(2.3.1) Definition. *A quasicoherent (dg)-sheaf on a dg-scheme X is a sheaf $\mathcal{F}^\bullet$ of $\mathcal{O}_X^\bullet$-dg-modules such that every $\mathcal{F}^i$ is quasicoherent over $\mathcal{O}_{X^0}$. If X is of finite* *type, we shall say that a quasicoherent sheaf $\mathcal{F}^\bullet$ is coherent, if each $\mathcal{F}^i$ is coherent over $\mathcal{O}_{X^0}$ and if $\mathcal{F}^\bullet$ is bounded above, i.e., $\mathcal{F}^i = 0$ for $i \gg 0$.*

If $\mathcal{F}$ is a quasicoherent dg-sheaf on X, we have graded sheaves $\mathcal{F}_\#$ on $X_\#$ and $\underline{H}^\bullet(\mathcal{F}^\bullet)$ on X_h.

Morphisms of quasicoherent dg-sheaves $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ and homotopies between such morphisms are defined in the obvious way, cf. (2.1).

A morphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is called a quasiisomorphism, if the induced morphism of graded sheaves $\underline{H}^\bullet(\mathcal{F}^\bullet) \to \underline{H}^\bullet(\mathcal{G}^\bullet)$ is an isomorphism.

We denote by $\mathcal{DQC}oh_X$ (resp. $\mathcal{DQC}oh_X^-$) the derived category of quasicoherent dg-sheaves (resp. of bounded above quasicoherent dg-sheaves) on X. Its objects are dg-sheaves of the described kind and morphisms are obtained by first passing to homotopy classes of morphisms and then localizing the resulting category by quasi-isomorphisms. Similarly, if X is of finite type, we have $\mathcal{DC}oh_X$, the derived category of coherent dg-sheaves. These are triangulated categories naturally associated to X.

If S is an ordinary scheme, by a graded vector bundle we mean a graded sheaf $E^\bullet$ of $\mathcal{O}_S$-modules such that each E^i is locally free of finite rank.

(2.3.2) Definition. *Let X be a dg-scheme of finite type. A dg-vector bundle on X is a coherent dg-sheaf $\mathcal{F}^\bullet$ such that locally, on the Zariski topology of X^0, the graded sheaf $\mathcal{F}^\bullet_\#$ of $\mathcal{O}^\bullet_{X,\#}$-modules is isomorphic to $\mathcal{O}^\bullet_{X,\#} \otimes_{\mathcal{O}_{X^0}} E^\bullet$ where $E^\bullet$ is a graded vector bundle on X^0 bounded from above.*

If X^0 is connected, the sequence $r = \{r_i\} = \{\mathrm{rk}(E^i)\}$ is uniquely defined by $\mathcal{F}^\bullet$ and is called the graded rank of $\mathcal{F}^\bullet$. We say that $\mathcal{F}^\bullet$ has bounded rank, if $r_i = 0$ for $i \ll 0$. In this case $\mathcal{F}^* = \mathcal{H}om_{\mathcal{O}^\bullet_X}(\mathcal{F}^\bullet, \mathcal{O}^\bullet_X)$ is again a dg-vector bundle of bounded rank.

(2.3.3) Notation. If $\mathcal{A}$ is a quasicoherent sheaf of $\mathbb{Z}_-$-graded dg-algebras on a dg-scheme X, then we have a dg-scheme $\mathrm{Spec}(\mathcal{A}) \to X$. If $\mathcal{F}^\bullet$ is a dg-vector bundle on X with $r_i(\mathcal{F}^\bullet) = 0$ for $i < 0$, then the symmetric algebra $S(\mathcal{F}^*)$ is $\mathbb{Z}_-$-graded and we denote $|\mathcal{F}^\bullet| = \mathrm{Spec}(S(\mathcal{F}^*))$. Similarly, if $\mathcal{F}^\bullet$ is a dg-vector bundle with $r_i = 0$ for $i > 0$, we write $|\mathcal{F}^*| = \mathrm{Spec}(S(\mathcal{F}^\bullet))$.

We now establish the existence of good resolutions of dg-sheaves by vector bundles. The following fact is well known (part (a) is in fact true for any scheme S which can be embedded as a closed subscheme into a smooth algebraic scheme, see [Fu] §B.8, and part (b) follows from it).

(2.3.4) Lemma. *Let S be a quasiprojective scheme. Then:*

(a) For any coherent sheaf $\mathcal{F}$ on S there exists a vector bundle E and a surjection $E \to \mathcal{F}$.

(b) For any quasicoherent sheaf $\mathcal{F}$ on S there exists an $\mathcal{O}_S$-flat quasicoherent sheaf $\mathcal{E}$ and a surjection $\mathcal{E} \to \mathcal{F}$.

(2.3.5) Proposition. *Let X be a quasiprojective dg-scheme and let $\mathcal{F}^\bullet$ be a quasicoherent dg-sheaf on X which is bounded from above. Then:*

(a) $\mathcal{F}^\bullet$ is quasiisomorphic to a quasicoherent dg-sheaf $\mathcal{E}^\bullet$, still bounded above and such that $\mathcal{E}_\#$ is flat over $\mathcal{O}_{X\#}$.

(b) If $\mathcal{F}^\bullet$ is coherent, it is quasiisomorphic to a vector bundle.

Proof. Standard inductive argument, using, at each step, Lemma 2.3.4 and a dg-module of the form $\mathcal{O}_X^\bullet \otimes_{\mathcal{O}_{X^0}} \mathcal{E}$ where $\mathcal{E}$ is a flat $\mathcal{O}_{X^0}$-sheaf or a vector bundle, cf. [Mc], §7.1.1.

(2.4) Derived tensor product. Let X be a dg-scheme. As $\mathcal{O}_X^\bullet$ is a sheaf of commutative dg-algebras, we can form the tensor product $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X^\bullet} \mathcal{G}^\bullet$ of any quasicoherent dg-sheaves. We will need the derived functor of the tensor product as well.

(2.4.1) Proposition. *Let $\mathcal{E}^\bullet, \mathcal{G}^\bullet$ be quasicoherent, bounded from above, dg-sheaves on a dg-scheme X and suppose that $\mathcal{E}_\#$ is flat over $\mathcal{O}_{X\#}$. Then we have the converging (Eilenberg-Moore) spectral sequence of sheaves on X^0*

$$E_2 = \mathcal{T}or_\bullet^{\underline{H}^\bullet(\mathcal{O}_X)}(\underline{H}^\bullet(\mathcal{E}^\bullet), \underline{H}^\bullet(\mathcal{G}^\bullet)) \quad \Rightarrow \quad \underline{H}^\bullet(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X^\bullet} \mathcal{G}^\bullet).$$

Assuming this proposition, we can make the following definition.

(2.4.2) Definition. *Let X be a quasiprojective dg-scheme and $\mathcal{F}^\bullet, \mathcal{G}^\bullet$ be quasicoherent dg-sheaves on X bounded above. The derived tensor product $\mathcal{F}^\bullet \otimes^L_{\mathcal{O}_X^\bullet} \mathcal{G}^\bullet$ is, by definition, the usual tensor product $\mathcal{E}^\bullet \otimes_{\mathcal{O}_X^\bullet} \mathcal{G}^\bullet$ where $\mathcal{E}^\bullet$ is any resolution of $\mathcal{F}^\bullet$ which is bounded above and #-flat.*

The existence of the resolution is given by (2.3.5), the independence of the resolution by (2.4.1).

(2.4.3) Proposition. *In the situation of (2.4.2), we have the two converging Eilenberg-Moore spectral sequences*

$$E_1 = \mathcal{T}or_\bullet^{\mathcal{O}_{X\#}}(\mathcal{F}_\#, \mathcal{G}_\#) \quad \Rightarrow \quad \underline{H}^\bullet(\mathcal{F}^\bullet \otimes^L_{\mathcal{O}_X^\bullet} \mathcal{G}^\bullet),$$

$$E_2 = \mathcal{T}or_\bullet^{\underline{H}^\bullet(\mathcal{O}_X)}(\underline{H}^\bullet(\mathcal{F}^\bullet), \underline{H}^\bullet(\mathcal{G}^\bullet) \quad \Rightarrow \quad \underline{H}^\bullet(\mathcal{F}^\bullet \otimes^L_{\mathcal{O}_X^\bullet} \mathcal{G}^\bullet).$$

Propositions 2.4.1-3 are obtained by globalizing the known statements about dg-modules over a dg-algebra, see, e.g., [Mc] §7.1.1. Compared to *loc. cit.* however, our class of algebras is more restricted and our class of resolutions is more general, so we indicate the main steps.

Let A be a $\mathbb{Z}_-$-graded commutative dg-algebra and P, Q be two dg-modules over A bounded above. Then we have an *ad hoc* definition of the derived tensor product based on the bar-resolution

$$\mathrm{Bar}_A(P) = \Big\{ ... \to A \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} P \to A \otimes_{\mathbb{K}} P \Big\} \xrightarrow{qis} P. \tag{2.4.4}$$

More precisely, $\mathrm{Bar}_A(P)$ is the total complex of the double complex inside the braces; denote by $\mathrm{Bar}^i_A(P)$, $i \leq 0$ the ith column of this double complex, i.e., $A^{\otimes(i+1)} \otimes P$. This resolution satisfies the following properties:

(2.4.5) Proposition. *(a)* $\mathrm{Bar}_A(P)_\#$ *is a free* $A_\#$*-module.*

(b) If $P \to P'$ *is a quasiisomorphism of dg-modules bounded from above, then* $\mathrm{Bar}_A(P) \to \mathrm{Bar}_A(P')$ *is a quasiisomorphism.*

(c) $H^\bullet(\mathrm{Bar}^i_A(M)) = \mathrm{Bar}^i_{H^\bullet(A)}(H^\bullet(M))$.

Proof. (a) is obvious; (b) follows by a spectral sequence argument (legitimate since the complexes are bounded from above) and (c) follows from the Künneth formula.

We define the ad hoc derived tensor product to be

$$P \boxtimes_A Q = \mathrm{Bar}_A(P) \otimes_A Q = \Big\{ ... \to P \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} Q \to P \otimes_{\mathbb{K}} Q \Big\}. \tag{2.4.6}$$

As before, this is in fact really the total complex of a double complex whose vertical differential is induced by d_P, d_A, d_Q and the horizontal one by the structure of A-modules on P, Q. So the standard spectral sequence of this double complex gives us the first Eilenberg-Moore spectral sequence in the form

$$E_1 = \mathrm{Tor}^{A_\#}_\bullet(P_\#, Q_\#) \quad \Rightarrow \quad H^\bullet(P \boxtimes_A Q). \tag{2.4.7}$$

(2.4.8) Corollary. *If* $F \to P$ *is a quasiisomorphism with* F *bounded above and* $F_\#$ *flat over* $A_\#$*, then* $P \boxtimes_A Q$ *is quasiisomorphic to the usual tensor product* $F \otimes_A Q$*. In particular,* $F \otimes_A Q$ *is independent on the choice of a* $\#$*-flat resolution* $F \to P$*.*

Part (c) of Proposition 2.4.5 gives the second Eilenberg-Moore spectral sequence in the form

$$E_2 = \mathrm{Tor}^{H^\bullet(A)}_\bullet(H^\bullet(P), H^\bullet(Q)) \quad \Rightarrow \quad H^\bullet(P \boxtimes_A Q). \tag{2.4.9}$$

(2.4.10) Corollary. *If* $P_\#$ *is flat over* $A_\#$*, then we have a spectral sequence converging to the ordinary tensor product*

$$E_2 = \mathrm{Tor}^{H^\bullet(A)}_\bullet(H^\bullet(P), H^\bullet(Q)) \quad \Rightarrow \quad H^\bullet(P \otimes_A Q).$$

Proposition 2.4.1 follows from (2.4.10) by gluing together the spectral sequences corresponding to $A = \Gamma(U, \mathcal{O}^\bullet_X)$ for affine open $U \subset X^0$. At the same time, we get the second spectral sequence in (2.4.3). As for the first spectral sequence, it follows by gluing together the spectral sequence obtained from (2.4.7), (2.4.8) and the definition of $\otimes^L$.

(2.5) Dg-manifolds and tangent complexes.

(2.5.1) Definition. *A dg-scheme M is called smooth (or a dg-manifold) if the following conditions hold:*

(a) M^0 is a smooth algebraic variety.

(b) Locally on Zariski topology of M^0, we have an isomorphism of graded algebras

$$\mathcal{O}^\bullet_{M^\#} \quad \simeq \quad S_{\mathcal{O}_{M^0}}\Big(Q^{-1} \oplus Q^{-2} \oplus ...\Big),$$

where Q^{-i} are vector bundles (of finite rank) on M^0.

An equivalent way of expressing (b) is that every truncation $\mathcal{O}^{\geq -d}_{M^\#}$ is locally, on M^0 isomorphic to the similar truncation of a free graded-commutative $\mathcal{O}_{M^0}$-algebra with finitely many generators in each degree.

The graded vector bundle $Q^\bullet_M = \bigoplus_{i\leq -1} Q^i_M$ from (b) can be defined globally as the bundle of primitive elements:

$$Q^\bullet_M = \mathcal{O}^{\leq -1}_M / (\mathcal{O}^{\leq -1}_M)^2, \tag{2.5.2}$$

but we do not have a natural embedding $Q^\bullet_M \to \mathcal{O}^\bullet_M$.

The *dimension* of a dg-manifold M is the sequence $\dim(M) = \{d_i(M)\}_{i\geq 0}$ where

$$d_0(M) = \dim(M^0), \quad d_i(M) = \mathrm{rk}(Q^{-i}_M), i > 0. \tag{2.5.3}$$

The *cotangent dg-sheaf* $\Omega^{1\bullet}_M$ of M is defined, as in the commutative case, as the target of the universal derivation $\delta : \mathcal{O}^\bullet_M \to \Omega^{1\bullet}_M$. The proof of the following proposition is standard.

(2.5.4) Proposition. *$\Omega^{1\bullet}_M$ is $\mathbb{Z}_-$-graded coherent sheaf, which is a vector bundle of rank $\{r_i = d_{-i}(M)\}_{i\leq 0}$.*

We have the *tangent dg-sheaf* $T^\bullet M$ defined as usual via derivations

$$T^\bullet M = \mathcal{D}er_{\mathbb{K}}(\mathcal{O}^\bullet_M, \mathcal{O}^\bullet_M) = \mathcal{H}om_{\mathcal{O}^\bullet_M}(\Omega^{1\bullet}_M, \mathcal{O}^\bullet_M). \tag{2.5.5}$$

This is a quasicoherent sheaf of dg-Lie algebras on M. Its differential is given by the commutator with the differential in $\mathcal{O}^\bullet_M$. It is coherent if and only if $d_i(M) = 0$ for $i \gg 0$. Further if $x \in M(\mathbb{K}) = \pi_0(M)(\mathbb{K})$, is a $\mathbb{K}$-point, the *tangent dg-space* to M at x is defined by

$$T^\bullet_x M = \mathrm{Der}_{\mathbb{K}}(\mathcal{O}^\bullet_M, \mathbb{K}_x) = T^\bullet M \otimes_{\mathcal{O}^\bullet_M} \mathbb{K}_x. \tag{2.5.6}$$

Here $\mathbb{K}_x$ is the 1-dimensional $\mathcal{O}^\bullet_M$-dg-module corresponding to x. Similarly, we can define the tangent space at any $\mathbb{F}$-point of $\pi_0(M)$, where $\mathbb{K} \subset \mathbb{F}$ is a field extension. We will sometimes use the following suggestive "topological" notation:

$$\pi_i(M, x) = H^{-i}(T^\bullet_x M), i < 0. \tag{2.5.7}$$

One justification of it is given by the following remark.

(2.5.8) Proposition. *Let $\mathbb{F}$ be an extension of $\mathbb{K}$ and x be an $\mathbb{F}$-point of x. Any choice of a formal coordinate system on M near x gives rise to a structure of a homotopy Lie algebra* [St2] *on the shifted tangent dg-space $T^\bullet_x M[-1]$. In particular, at the level of cohomology we have well defined "Whitehead products"*

$$[-,-] : \pi_i(M,x) \otimes_{\mathbb{F}} \pi_j(M,x) \to \pi_{i+j-1}(M,x)$$

making $\pi_{\bullet+1}(M,x)$ into a graded Lie algebra over $\mathbb{F}$.

This statement, see, e.g., [Ka1, Prop. 1.2.2], is in fact equivalent to the very definition of a homotopy Lie algebra and should be regarded as being as old as this definition. More precisely [St2], if $\mathfrak{g}^\bullet$ is a graded vector space, a structure of a homotopy Lie algebra on $\mathfrak{g}^\bullet$ is the same as a continuous derivation D of the completed symmetric algebra $\widehat{S}^\bullet(\mathfrak{g}^*[-1])$ satisfying $D^2 = 0$ (so $(\widehat{S}^\bullet(\mathfrak{g}^*[-1]), D)$ serves as the cochain complex of $\mathfrak{g}^\bullet$). If we take $\mathfrak{g}^\bullet = T^\bullet_x M[-1]$, then the completed local ring $\widehat{\mathcal{O}}^\bullet_{M,x}$ with its natural differential d, serves as a cochain complex for $\mathfrak{g}^\bullet$: a choice of formal coordinates identifies it with $\widehat{S}^\bullet(\mathfrak{g}^*[-1])$.

These two Lie algebra structures (one on the tangent sheaf, the other on its shifted fiber) will be related in (2.7.9).

As usual, any morphism $f : M \to N$ of dg-manifolds induces a morphism of coherent dg-sheaves $d^*f : f^*\Omega^{1\bullet}_N \to \Omega^{1\bullet}_M$ and for any $x \in M(\mathbb{F}) = \pi_0(M)(\mathbb{F})$, a morphism of complexes $d_x f : T^\bullet_x M \to T^\bullet_{f(x)} N$. These morphisms of complexes fit together into a morphism of quasicoherent dg-sheaves $df : T^\bullet M \to f^* T^\bullet N$.

The equivalence (i)⇔(ii) in the following proposition (see [Ka1, Prop. 1.2.3]) can be seen as an analog of the Whitehead theorem in topology.

(2.5.9) Proposition. *(a) Let $f : M \to N$ be a morphism of dg-manifolds. Then the following conditions are equivalent:*

(i) f is a quiasiisomorphism.

(ii) The morphism of schemes $\pi_0(f) : \pi_0(M) \to \pi_0(N)$ is an isomorphism, and for any field extension $\mathbb{K} \subset \mathbb{F}$ and any $\mathbb{F}$-point x of M the differential $d_x f$ induces an isomorphism $\pi_i(M,x) \to \pi_i(N, f(x))$ for all $i < 0$.

(iii) $\pi_0(f)$ is an isomorphism and $d^*f : f^*\Omega^{1\bullet}_N \to \Omega^{1\bullet}_M$ is a quasiisomorphism of coherent dg-sheaves on M.

(b) If any of these conditions is satisfied, then $df : T^\bullet M \to f^* T^\bullet N$ is a quasi-isomorphism.

Proof. We first establish the equivalences in (a)

(ii)⇒(i) To show that f is an isomorphism, it is enough to prove that for any field extension $\mathbb{F} \supset \mathbb{K}$ and any $x \in \pi_0(M)(\mathbb{F})$ the map $\hat{f}^*\widehat{\mathcal{O}}^\bullet_{N,f(x)} \to \widehat{\mathcal{O}}^\bullet_{M,x}$ which f induces on the completed local dg-algebras, is a quasiisomorphism. For that, notice that $\widehat{\mathcal{O}}^\bullet_{M,x}$ has a filtration whose quotients are the symmetric powers of the cotangent dg-space $T^*_x M$. So if f gives a quasiisomorphism of tangent dg-spaces, we find that $\hat{f}^*$ induces quasiisomorphisms on the quotients of the natural filtrations. So the proof is accomplished by invoking a spectral sequence argument, which is legitimate (i.e., the spectral sequences converge) because the dg-algebras in question are $\mathbb{Z}_{\leq 0}$-graded.

(iii)⇒(ii) Since $d^* f$ is a quasiisomorphism of dg-vector bundles, it induces, by taking the tensor product with $\mathbb{F}_x$, a quasiisomorphism on the fiber at each $x \in M(\mathbb{K})$. The fibers of $\Omega^{1\bullet}_M$ and $f^*\Omega^{1\bullet}_N$ at x are just the complexes dual to $T^\bullet_x M$ and $T^\bullet_{f(x)} N$; in particular, they are finite-dimensional in each degree. Thus the dual map, which is $d_x f$, is a quasiisomorphism as well.

(i)⇒(iii) It is enough to show that the morphism $d^* f : f^{-1}\Omega^{1\bullet}_N \to \Omega^{1\bullet}_M$ is a quasiisomorphism. Indeed, knowing this, (iii) is obtained by applying the functor $- \otimes_{f^{-1}\mathcal{O}^\bullet_N} \mathcal{O}^\bullet_M$ (to pass from f^{-1} to f^*) and invoking the Eilenberg-Moore spectral sequence and the fact that $f^{-1}\mathcal{O}^\bullet_N \to \mathcal{O}^\bullet_M$ is a quasiisomorphism. Since we can work locally, all we need to prove is a statement about dg-algebras. We call a commutative dg-algebra smooth if its spectrum is an affine dg-manifold.

(2.5.10) Lemma. *If $\phi : A \to B$ is a quasiisomorphism of smooth $\mathbb{Z}_-$-graded commutative dg-algebras, then $d\phi : \Omega^{1\bullet}_A \to \Omega^{1\bullet}_B$ is a quasiisomorphism of complexes.*

Proof. This is a standard application of the theory of Harrison homology, cf. [Lo, §4.2.10]. The Harrison chain complex is

$$\mathrm{Harr}_\bullet(A, A) = \mathrm{FCoLie}(A[-1]) \otimes A$$

where FCoLie stands for the free graded coLie algebra generated by a graded vector space[2]. It satisfies the following properties:

(a) $\mathrm{Harr}_\bullet(A, A)$ is covariantly functorial in A and its dependence on A is exact (takes quasiisomorphisms to quasiisomorphisms).

(b) If A is smooth and $d_A = 0$, then $\mathrm{Harr}_\bullet(A, A)$ is quasiisomorphic to $\Omega^{1\bullet}_A$.

Part (a) is obvious from the tensor nature of the functor FCoLie. Part (b) can be proved in the same way as for ordinary (not dg) smooth algebras: by realizing $\mathrm{Harr}_\bullet(A, A)$ as indecomposable elements in the Hochschild complex and using the Hochschild-Kostant-Rosenberg theorem, see [Lo, Thm. 3.4.4].

This concludes the proof of part (a) of Proposition 2.5.9. To prove (b), it is enough to work locally, so to assume $M = \mathrm{Spec}(A)$ is affine. Then we view $d^* f : f^*\Omega^{1\bullet}_N \to \Omega^{1\bullet}_M$ as a morphism of dg-modules over A, and we know that is a quasiisomorphism. By further localizing on $\mathrm{Spec}(A^0)$, we can assume that A^0 is a local ring. Then, projective modules over A^0 being free, we have that both $f^*\Omega^{1\bullet}_N$

[2] Since the primitive elements in a free tensor coalgebra give the free coLie algebra, this description coincides with the more traditional one which gives the Harrison complex as the primitive elements in the Hochshild complex of A.

and $\Omega^{1\bullet}_M$ are quasifree dg-modules over A. Hence, by Corollary (2.1.5), the dual morphism to d^*f, i.e., $df : T^\bullet M \to f^*T^\bullet N$ is a quasiisomorphism as well.

(2.6) Existence of smooth resolutions. A morphism of dg-schemes $f : X \to Y$ will be called a closed embedding, if $f_0 : X^0 \to Y^0$ is a closed embedding of schemes and the structure morphism of sheaves of dg-algebras $f_0^*\mathcal{O}^\bullet_N \to \mathcal{O}^\bullet_M$ is surjective.

(2.6.1) Theorem. *(a) For any quasiprojective dg-scheme X there is a dg-manifold M and a quasiisomorphic closed embedding $X \hookrightarrow M$.*

(b) Given any two embeddings $X \hookrightarrow M$, $X \hookrightarrow N$ as in (a), they can be complemented by quasiisomorphic closed embeddings $M \hookrightarrow L$, $N \hookrightarrow L$ for some dg-manifold L so that the natural square is commutative.

Proof. (a)⇒(b). Given M, N, we set

$$Y = M \cup_X N = \left(M^0 \cup_{X^0} N^0,\ \mathcal{O}^\bullet_M \times_{\mathcal{O}^\bullet_X} \mathcal{O}^\bullet_N\right).$$

Then we have a diagram as required except that Y may be not smooth. To amend this, it suffices to embed Y into a dg-manifold L as in (a).

(a) This is a version of the standard fact asserting the existence of a free resolution for a $\mathbb{Z}_-$-graded dg-algebra, see, e.g., [BG] §4.7. If X is affine, then this fact indeed implies what we need.

In the general case, let $X = (X^0, \mathcal{O}^\bullet_X)$ be given. As X^0 is a quasiprojective scheme, we can choose its embedding into a P^n as a locally closed subscheme. Take M^0 to be an open subset in P^n such that X^0 is closed in M^0. We then construct $\mathcal{O}^\bullet_M$ by induction as the union of sheaves of dg-subalgebras

$$\mathcal{O}_{M^0} = \mathcal{O}^\bullet_0 \subset \mathcal{O}^\bullet_1 \subset \mathcal{O}^\bullet_2 ...$$

such that:

(1) $\mathcal{O}^\bullet_i$ is obtained from $\mathcal{O}^\bullet_{i-1}$ by adding a vector bundle of generators in degree $(-i)$.

(2) We have a compatible system of algebra morphisms $p^{(i)} : \mathcal{O}^\bullet_i \to \mathcal{O}^\bullet_X$ so that each $p^{(i)}$ is bijective on $\underline{H}^j$ for $-i+1 \leq j \leq 0$ and surjective on the sheaf of jth cocycles for $-i \leq j \leq 0$.

The following elementary lemma shows that surjectivity on cocycles implies surjectivity on graded components and therefore the map we will construct in this way will be a closed embedding.

(2.6.2) Lemma. *Let $\phi : C^\bullet \to D^\bullet$ be a quasiisomorphism of complexes of vector spaces which is surjective on cocycles. Then each $\phi_i : C^i \to D^i$ is surjective.*

The inductive construction of the $\mathcal{O}^\bullet_i$ follows the standard pattern of "imitating the procedure of attaching cells to kill homotopy groups" ([Q2], p. 256, see also [BG], §4.7) except that we use Lemma 2.3.4 to produce a vector bundle of generators. We leave the details to the reader.

(2.7) Smooth morphisms. We now relativize the above discussion.

(2.7.1) Definition. *Let M, N be dg-schemes of finite type. A morphism $f : M \to N$ is called smooth, if the following two conditions hold:*

(a) The underlying morphism $f_0 : M^0 \to N^0$ of ordinary schemes is smooth.

(b) Locally, on the Zariski topology of M^0, we have an isomorphism of graded algebras

$$\mathcal{O}^\bullet_{M\#} \simeq S_{M^0}(Q^\bullet) \otimes f_0^* \mathcal{O}^\bullet_{N\#},$$

where $Q^\bullet = \bigoplus_{i \le -1} Q^i$ is a graded vector bundle on M^0.

As before, for a smooth morphism we can always globally define the graded bundle

$$Q^\bullet_{M/N} \quad = \left(\mathcal{O}^{\le -1}_{M\#} / (\mathcal{O}^{\le -1}_{M\#})^2 \right) \otimes_{f_0^* \mathcal{O}^\bullet_{N\#}} \mathcal{O}^\bullet_M, \tag{2.7.2}$$

but it embeds into $\mathcal{O}^\bullet_M$ only locally. We also have the relative cotangent dg-sheaf $\Omega^{1\bullet}_{M/N}$ which is a $\mathbb{Z}_-$-graded vector bundle and the relative tangent dg-sheaf

$$T^\bullet(M/N) = \mathcal{D}er^\bullet_{f_0^{-1}\mathcal{O}^\bullet_N}(\mathcal{O}^\bullet_M, \mathcal{O}^\bullet_M) = \mathcal{H}om_{\mathcal{O}^\bullet_M}(\Omega^{1\bullet}_{M/N}, \mathcal{O}^\bullet_M). \tag{2.7.3}$$

This is a quasicoherent dg-sheaf.

Let $x : N \to M$ be a section of f (i.e., an N-point of an N-dg-scheme M). Then we define the tangent dg-space (or bundle) to M/N at (or along) x as

$$T^\bullet_x(N/M) = x^* T^\bullet(N/M) = x_0^* T^\bullet(N/M) \otimes_{x_0^* \mathcal{O}^\bullet_M} \mathcal{O}^\bullet_N. \tag{2.7.4}$$

This is a quasicoherent dg-sheaf on N.

(2.7.5) Remarks. (a) As in (2.5.7), one can show that $T^\bullet_x(M/N)[-1]$ is a "sheaf of homotopy Lie algebras" on N; in particular, its cohomology $\underline{H}^\bullet T^\bullet_x(M/N)[-1]$ is naturally a sheaf of graded Lie algebras. Indeed, the role of the "cochain complex" of $T^\bullet_x(M/N)[-1]$ is played by $\widehat{\mathcal{O}}^\bullet_{M,x}$ the completion of $\mathcal{O}^\bullet_M$ along the subscheme $x(N)$.

(b) Globally it may be impossible to identify $\widehat{\mathcal{O}}^\bullet_{M,x}$ with the symmetric algebra of $T^\bullet_x(M/N)[-1]$ and the corresponding obstruction gives rise to another Lie algebra-type structure, present even when M, N are ordinary (not dg) schemes. More precisely, the obstruction to splitting the second infinitesimal neighborhood gives rise to a version of the Atiyah class:

$$\alpha \in H^1(N^0, \mathcal{H}om(S^2 T^\bullet_x(M/N)[-1], T^\bullet_x(M/N)[-1])$$

which formally satisfies the Jacobi identity, as an element of an appropriate operad. This generalizes the main observation of [Ka2, Thm 3.5.1], which corresponds to the case when $N = X$ is an ordinary manifold, $M = X \times X$ and x is the diagonal map.

The following smoothing statement can be regarded as a rudiment of a closed model structure in the category of dg-schemes.

(2.7.6) Theorem. *Let $f : M \to N$ be any morphism of quasiprojective dg-schemes. Then f can be factored as $M \overset{i}{\hookrightarrow} \widetilde{M} \overset{\widetilde{f}}{\to} N$, where i is a quasiisomorphic closed embedding and $\widetilde{f}$ is smooth. Any two such factorizations can be included into a third one.*

Proof. We embed M_0 into $N_0 \times P^n$. Then take for $\widetilde{M}_0$ an open subset in $N_0 \times P^n$ such that M_0 is closed in $\widetilde{M}_0$. Then the procedure is the same as outlined in (2.6.1) for $N = pt$.

A diagram $M \overset{i}{\hookrightarrow} \widetilde{M} \overset{\widetilde{f}}{\to} N$ as in (2.7.6) will be called a smooth resolution of $f : M \to N$.

The following is a relative version of a part of Proposition 2.5.9, proved in the same way.

(2.7.7) Proposition. *Let*

$$\begin{array}{ccc} M_1 & \xrightarrow{q} & M_2 \\ f_1 \searrow & & \swarrow f_2 \\ & N & \end{array}$$

be a commutative triangle with $f_i, i = 1, 2$ smooth and q a quasiisomorphism. Then $dq : T^\bullet(M_1/N) \to q^*T^\bullet(M_2/N)$ and $d^*q : q^*\Omega^{1\bullet}_{M_2/N} \to \Omega^{1\bullet}_{M_1/N}$ are quasiisomorphisms.

(2.7.8) Definition. *The derived relative tangent complex of a morphism $f : M \to N$ of quasiprojective dg-schemes is, by definition,*

$$RT^\bullet(M/N) = T^\bullet(\widetilde{M}/N),$$

where $\widetilde{f} : \widetilde{M} \to N$ is any smooth resolution of f. This is a sheaf of dg-Lie algebras on $\widetilde{M}$.

Proposition 2.7.7 guarantees that $RT^\bullet(M/N)$ is well defined up to quasiisomorphism.

(2.7.9) Example. Let $M = \{x\}$ be a $\mathbb{K}$-point of N and f be the embedding of this point. Then the derived relative tangent complex $RT^\bullet(\{x\}/N)$ is quasiisomorphic to the shifted tangent complex $T^\bullet_x N[-1]$. This can be seen by taking for $\widetilde{M}$ the spectrum of a Koszul resolution of $\mathbb{K}_x$ on an affine open dg-submanifold $U \subset N$ containing x. The presence of the Lie bracket on $T^\bullet(\widetilde{M}/N) = RT^\bullet(\{x\}/N)$ provides an alternative explanation of the presence of a homotopy Lie algebra structure on $T_x^\bullet N[-1]$.

(2.8) Derived fiber products. Let $f_i : M_i \to N$ be morphisms of dg-schemes, $i = 1, 2$. The fiber product $M_1 \times_N M_2$ is defined as follows. First, we form the fiber product of underlying ordinary schemes:

$$\begin{tikzcd} M_1^0 \times_{N^0} M_2^0 \arrow[r, "g_{2,0}"] \arrow[d, "g_{1,0}"'] & M_2^0 \arrow[d, "f_{2,0}"] \\ M_1^0 \arrow[r, "f_{1,0}"'] & N^0 \end{tikzcd}$$

and define

$$M_1 \times_N M_2 \quad = \quad \left(M_1^0 \times_{N^0} M_2^0,\ g_{2,0}^{-1}\mathcal{O}^\bullet_{M_2} \otimes_{(f_{1,0}g_{1,0})^{-1}\mathcal{O}^\bullet_N} g_{1,0}^{-1}\mathcal{O}^\bullet_{M_1} \right),$$

so that we have the natural square

$$\begin{tikzcd} M_1 \times_N M_2 \arrow[r, "g_2"] \arrow[d, "g_1"'] & M_2 \arrow[d, "f_2"] \\ M_1 \arrow[r, "f_1"'] & N \end{tikzcd} \tag{2.8.1}$$

The following fact is clear.

(2.8.2) Proposition. *If f_2 is a smooth morphism, then so is g_1.*

The fiber (or preimage) is a particular case of this construction. More precisely, let $f : M \to N$ be a morphism and $y \in N(\mathbb{K})$ be a point. The fiber $f^{-1}(y)$ is the fiber product $M \times_N \{y\}$. If f is a smooth morphism, then $f^{-1}(y)$ is a dg-manifold. Suppose further that M, N and f are all smooth. Then we have the Kodaira-Spencer map

$$\varkappa : T_y^\bullet N[-1] \to R\Gamma(f^{-1}(y), T^\bullet f^{-1}(y)) \tag{2.8.3}$$

which is, as in the standard case, obtained from the short exact sequence

$$0 \to T^\bullet(M/N) \to T^\bullet M \to f^* T^\bullet N \to 0$$

by tensoring over $\mathcal{O}^\bullet_M$ with $\mathcal{O}^\bullet_{f^{-1}(y)}$ and using the adjunction.

(2.8.4) Remark. Note that both the source and target of $\varkappa$ possess a homotopy Lie algebra structure: the source by Proposition 2.5.8, the target as the direct image of a sheaf of dg-Lie algebras. In fact, it can be shown that $\varkappa$ is naturally a homotopy morphism of homotopy Lie algebras. In particular, the graded Lie algebra $\pi_{\bullet+1}(N,y)$ acts on the hypercohomology space $\mathbb{H}^\bullet(f^{-1}(y),\mathcal{O}^\bullet)$ in a way remindful of the monodromy action of a fundamental group. We postpone further discussion to a more detailed exposition of the basics of the theory, to be completed at a future date.

Note that a smooth morphism is flat (this is proved in the same way as for the case of usual schemes). Therefore Proposition 2.4.1 implies the following.

(2.8.5) Proposition. *Suppose that in the fiber product diagram (2.8.1) the morphism f_1 (and hence g_2) is smooth. Then we have a converging (Eilenberg-Moore) spectral sequence of quasicoherent sheaves of graded algebras on M^0:*

$$E_2 = \mathcal{T}or_\bullet^{(f_{10}g_{10})^{-1}\underline{H}^\bullet(\mathcal{O}^\bullet_N)}\big(g_1^{-1}\underline{H}^\bullet(\mathcal{O}^\bullet_{M_1}),\ g_2^{-1}\underline{H}^\bullet(\mathcal{O}_{M_2})\big) \quad\Rightarrow\quad \underline{H}^\bullet(\mathcal{O}^\bullet_{M_1\times_N M_2}).$$

(2.8.6) Definition. *The derived (or homotopy) fiber product $M_1\times^R_N M_2$ is defined as $\widetilde{M}_1\times_N M_2$ where $\widetilde{f}_1:\widetilde{M}_1\to N$ is a smooth resolution of the morphism f_1.*

(2.8.7) Lemma. *The definition of $M_1\times^R_N M_2$ is independent, up to quasiisomorphism, of the choice of a smooth resolution of f.*

Proof. It is enough to show that whenever we have a diagram

$$\begin{tikzcd} \widetilde{M}_1 \arrow[rr, hook, "q"] \arrow[dr, "\widetilde{f}_1"'] & & \widetilde{M}'_1 \arrow[dl, "\widetilde{f}'_1"] \\ & N & \end{tikzcd}$$

with q a quasiisomorphism and $\widetilde{f}_1,\widetilde{f}'_1$ smooth, the induced morphism $\widetilde{M}_1\times_N M_2\to\widetilde{M}'_1\times_N M_2$ is a quasiisomorphism. For this notice that we have a morphism from the Eilenberg-Moore spectral sequence calculating $\underline{H}^\bullet\big(\mathcal{O}^\bullet_{\widetilde{M}_1\times_N M_2}\big)$ to the similar sequence calculating $\underline{H}^\bullet\big(\mathcal{O}^\bullet_{\widetilde{M}'_1\times_N M_2}\big)$ and this morphism is an isomorphism on E_2 terms.

We can now formulate the final form of the Eilenberg-Moore spectral sequences for the derived fiber products.

(2.8.8) Proposition. *Suppose a square of quasiprojective dg-schemes*

$$\begin{tikzcd} M \arrow[r, "g_2"] \arrow[d, "g_1"'] & M_2 \arrow[d, "f_2"] \\ M_1 \arrow[r, "f_1"'] & N \end{tikzcd}$$

is homotopy cartesian, i.e., the natural morphism $m \to M_1 \times^R_N M_2$ *is a quasiisomorphism. Then we have the two convergent spectral sequences of sheaves of algebras on* M^0*:*

$$E_1 = \mathcal{T}or_\bullet^{(f_{1,0}g_{1,0})^{-1}(\mathcal{O}^\bullet_{N\#})}\big(g_1^{-1}\mathcal{O}^\bullet_{M_1\#},\ g_2^{-1}\mathcal{O}^\bullet_{M_2\#}\big) \quad\Rightarrow\quad \underline{H}^\bullet(\mathcal{O}^\bullet_M),$$

$$E_2 = \mathcal{T}or_\bullet^{(f_{1,0}g_{1,0})^{-1}(\underline{H}^\bullet(\mathcal{O}^\bullet_N))}\big(g_1^{-1}\underline{H}^\bullet(\mathcal{O}^\bullet_{M_1}),\ g_2^{-1}\underline{H}^\bullet(\mathcal{O}^\bullet_{M_2})\big) \quad\Rightarrow\quad \underline{H}^\bullet(\mathcal{O}^\bullet_M).$$

(2.8.9) Remark. More generally, one can define the derived fiber product for any morphisms $f_i : M_i \to N$ of arbitrary dg-schemes (not necessarily quasiprojective or of finite type). But we need to assume that at least one of the f_i can be quasiisomorphically replaced by a #-flat morphism $\widetilde{M}_i \to N$. This is the case, for example, when f_i is an affine morphism (use the relative bar-resolution).

(2.8.10) Examples *(a) Derived intersection.* If Y, Z are closed subschemes of a quasiprojective dg-scheme X, then the derived intersection $Y \cap^R Z$ is defined as the derived fiber product of Y and Z over X. If X, Y, Z are ordinary (not dg) schemes, then the cohomology sheaves of $\mathcal{O}^\bullet_{Y\cap^R Z}$ are the $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$, see [Kon], n. (1.4.2).

(b) Homotopy fibers. Given any morphism $f : M \to N$ of quasiprojective dg-schemes and any point $y \in N(\mathbb{K})$, we have the homotopy fiber $Rf^{-1}(y) := \widetilde{f}^{-1}(y)$ where $\widetilde{f}$ is a smooth resolution of f. Note that for N smooth (and f arbitrary) we always have the derived Kodaira-Spencer map

$$R\varkappa : T^\bullet_y N[-1] \to R\Gamma(Rf^{-1}(y), T^\bullet)$$

and Remark 2.8.4 applies to this situation as well.

(c) The loop space. Consider the particular case of (b), namely $M = \{y\}$ and $f = i_y$ being the embedding. Using the topological analogy, it is natural to call the homotopy fiber $Ri_y^{-1}(y)$ the loop space of N at y and denote it $\Omega(N, y)$. This dg-scheme has only one $\mathbb{K}$-point, still denoted y ("the constant loop"). As for the tangent space at this point, we have $T^\bullet_y\Omega(N, y) = T^\bullet_y N[-1]$ and the derived Kodaira-Spencer map for i_y is the identity.

By going slightly beyond the framework of this paper, we can make the analogy with the usual loop space even more pronounced. Namely, consider the $\mathbb{Z}_+$-graded dg-algebra $\Lambda[\xi]$, $\deg(\xi) = 1$, in other words, $\Lambda[\xi] = H^\bullet(S^1, \mathbb{K})$ is the topological cohomology of the usual circle. Let us formally associate to this algebra the dg-scheme $\mathbb{S} = \mathrm{Spec}(\Lambda[\xi])$ ("dg-circle"). It has a unique $\mathbb{K}$-point which we denote e. Then we can identify $\Omega(N, y)$ with the internal Hom in the category of pointed dg-schemes

$$\Omega(N, y) = \underline{\mathrm{Hom}}\big((\mathbb{S}, e), (N, y)\big),$$

similarly to the usual definition of the loop space.

Further, the fact that the usual loop space is a group up to homotopy, has the following analog, cf. [Q1]. Let $\Pi \to N$ be a smooth quasiisomorphic replacement of $i_y : \{y\} \to N$, see (2.7.9). Then we have a groupoid $\mathfrak{G}$ in the category of dg-schemes with

$$\mathrm{Ob}\ \mathfrak{G} = \Pi \sim \{\mathrm{pt}\}, \quad \mathrm{Mor}\ \mathfrak{G} = \Pi \times_N \Pi \sim \Omega(N, y).$$

This group-like structure on $\Omega(N, y)$ provides still another explanation of the fact that its tangent space $T_y^\bullet N[-1]$ is a homotopy Lie algebra.

3. A finite-dimensional model

(3.1) The problem. Our goal in this paper is to construct, in the situation of (1.1), a dg-manifold $RSub_h(\mathcal{F})$ satisfying the conditions (0.3.1) and (0.3.2). In this section we consider a finite-dimensional analog of this problem. Namely, let A be a finite-dimensional associative algebra, M a finite-dimensional left A-module and $G_A(k,M)$ the A-Grassmannian, see (1.3). We want to construct a dg-manifold $RG_A(k,M)$ with the properties;

$$\pi_0\, RG_A(k,M) = G_A(k,M), \quad H^i T^\bullet_{[V]} RG_A(k,M) = \mathrm{Ext}^i_A(V, M/V). \tag{3.1.1}$$

As we will see later, the problem of constructing the derived Quot scheme can be reduced to this.

(3.2) Idea of construction. We first realize $G_A(k,M)$ inside the (noncompact) smooth variety $|\mathcal{H}om(A\otimes\widetilde{V},\widetilde{V})|$ as follows. The bundle $\mathcal{H}om(A\otimes\widetilde{V},\widetilde{V})$ has a canonical section s defined over the subscheme $G_A(k,M)$. This section is given by the A-action $A\otimes V\to V$ present on any submodule $V\subset M$. We embed $G_A(k,M)$ into $|\mathcal{H}om(A\otimes\widetilde{V},\widetilde{V})|$ as the graph of this section and will construct $RG_A(k,M)$ so that its underlying ordinary manifold is $|\mathcal{H}om(A\otimes\widetilde{V},\widetilde{V})|$. For this, we will represent the embedded $G_A(k,M)$ as the result of applying the following two abstract constructions.

(3.2.1) The space of actions. Let V be a finite-dimensional vector space. Then we have the subscheme $\mathrm{Act}(A,V)$ in the affine space $\mathrm{Hom}_{\mathbb{K}}(A\otimes_{\mathbb{K}} V, V)$ consisting of all A-actions (i.e., all A-module structures) on V. Note that we do not identify here two A-module structures which give isomorphic modules.

(3.2.2) The linearity locus. Let S be a scheme, and M,N be two vector bundles over S with A-actions in fibers. In other words, M,N are $\mathcal{O}_S\otimes_{\mathbb{K}} A$-modules which are locally free as $\mathcal{O}_S$-modules. Let also $f: M\to N$ be an $\mathcal{O}_S$-linear morphism. Its linearity locus is the subscheme

$$\mathrm{Lin}_A(f) = \Big\{ s\in S \Big| f_s : M_s \to N_s \quad \text{is} \quad A\text{-linear} \Big\}.$$

This is just the fiber product

$$\begin{array}{ccc}
\mathrm{Lin}_A(f) & \longrightarrow & S \\
\big\downarrow & & \big\downarrow f \\
|\mathcal{H}om_{A\otimes\mathcal{O}_S}(M,N)| & \longrightarrow & |\mathcal{H}om_{\mathcal{O}_S}(M,N)|
\end{array}$$

Let us apply the first construction to each fiber V of the bundle $\widetilde{V}$ on $G(k,M)$. We get the fibration

$$\mathrm{Act}(A,\widetilde{V}) \xrightarrow{q} G(k,M)$$

which is embedded into $|\mathcal{H}om(A\otimes\widetilde{V},\widetilde{V})|$. By construction, the pullback $q^*\widetilde{V}$ is a bundle of A-modules. Let also $\underline{M}$ be the trivial bundle of A-modules on $\mathrm{Act}(A,\widetilde{V})$ with fiber M. Then, we have the tautological morphism $f: q^*\widetilde{V}\to\underline{M}$ of vector bundles whose fiber over a point $([V],\alpha)\in\mathrm{Act}(A,\widetilde{V})$ is just the embedding $V\hookrightarrow M$.

(3.2.3) Proposition. *For this morphism f the scheme $\mathrm{Lin}_A(f)\subset\mathrm{Act}(A,\widetilde{V})$ coincides with $G_A(k,M)$ embedded into $\mathrm{Act}(A,\widetilde{V})\subset|\mathcal{H}om(A\otimes\widetilde{V},\widetilde{V})|$.*

Proof. Given a linear subspace $V\subset M$ and an A-action $\alpha: A\otimes V\to V$, the condition that the embedding $V\hookrightarrow M$ be A-linear precisely means that V is a submodule and α is the induced action.

Now the idea of constructing $RG_A(k,M)$ is to develop the derived analogs of the two constructions (3.2.1), (3.2.2) and apply them to the situation just described.

(3.3) The derived space of actions. Let us first analyze in more detail the (non-derived) construction $\mathrm{Act}(A,V)$. It can be defined for any (possibly infinite-dimensional) associative algebra A and a finite-dimensional vector space V. In this case we can apply the conventions of (2.2.3)(b) to the complex $F=\mathrm{Hom}(V,A\otimes V)$ and denote the affine scheme $|F^*|=\mathrm{Spec}\, S(F)$ by $|\mathrm{Hom}(A\otimes V,V)|$. The scheme $\mathrm{Act}(A,V)$ is the closed subscheme of $|\mathrm{Hom}(A\otimes V,V)|$, whose coordinate ring is $S(\mathrm{Hom}(V,A\otimes V))$ modulo the ideal expressing the associativity conditions. At the level of $\mathbb{K}$-points, $\mu: A\otimes V\to V$ is an action if and only if the map

$$\text{(3.3.1)}\quad \delta\mu: A\otimes A\otimes V\to V,\quad a_1\otimes a_2\otimes v\mapsto \mu(a_1a_2\otimes v)-\mu(a_1\otimes\mu(a_2\otimes v)),$$

vanishes. In this case $T_\mu\,\mathrm{Act}(A,V)$ is identified with the space of 1-cocycles in the bar-complex

$$\text{(3.3.2)}\qquad \mathrm{Hom}_{\mathbb{K}}(V,V)\to\mathrm{Hom}_{\mathbb{K}}(A\otimes V,V)\to\mathrm{Hom}_{\mathbb{K}}(A\otimes A\otimes V,V)\to ...$$

calculating $\mathrm{Ext}^\bullet_A(V,V)$.

(3.3.3) Remark. The reason that we get the space of 1-cocyles instead of the cohomology which is a more invariant object is that we do not identify isomorphic module structures. If we consider the quotient stack of $\mathrm{Act}(A,V)$ by $GL(V)$, then the tangent space to this stack at a point μ is a 2-term complex concentrated in degrees $0,-1$ and

$$H^iT^\bullet_\mu\Big(\mathrm{Act}(A,V)/GL(V)\Big)=\mathrm{Ext}^{i+1}_A(V,V),\quad i=0,-1.$$

Our aim in this subsection is to construct, for each finite-dimensional A, a (smooth) dg-manifold $\mathrm{RAct}(A,V)$ with $\pi_0 = \mathrm{Act}(A,V)$ and the tangent space at any $\mu\in\mathrm{Act}(A,V)$ having

$$H^iT^\bullet_\mu\,\mathrm{RAct}(A,V) = \begin{cases} T_\mu\,\mathrm{Act}(A,V), & i=0\\ \mathrm{Ext}^{i+1}_A(V,V), & i>0\end{cases} \tag{3.3.4}$$

The method of construction will be the standard approach of homological algebra, namely using free associative resolutions of A. This is similar to C. Rezk's approach [Re] to constructing "homotopy" moduli spaces for actions of an operad. More precisely, we will construct for any, possibly infinite-dimensional A, an affine dg-scheme $\mathrm{RAct}(A,V)$ whose coordinate algebra is free commutative, and will show that for $\dim(A)<\infty$, we can choose a representative with finitely many free generators in each degree, so that we have a dg-manfold.

Notice first that the construction of $\mathrm{Act}(A,V)$ in the beginning of this subsection can be carried through for any $\mathbb{Z}_-$-graded associative dg-algebra (with A^i possibly infinite-dimensional) and V a finite-dimensional vector space (which we think as being graded, of degree 0). As in the ungraded case, $\mathrm{Act}(A,V)$ is a closed dg-subscheme in $|\operatorname{Hom}(A\otimes V,V)|$ given by the ideal of associativity conditions, which is now a dg-ideal. The association $A\mapsto\mathrm{Act}(A,V)$ is functorial: a morphism of dg-algebras $f:A_1\to A_2$ gives rise to a morphism of dg-schemes $f^*:\mathrm{Act}(A_2,V)\to\mathrm{Act}(A_1,V)$.

Next, assume that $A=F(E^\bullet)$ is a free associative (tensor) algebra without unit generated by a $\mathbb{Z}_-$-graded vector space $E^\bullet$. Then, clearly, we have

$$\mathrm{Act}(F(E^\bullet),V) = |\operatorname{Hom}_{\mathbb{C}}(E^\bullet\otimes V,V)|, \tag{3.3.5}$$

as an action is uniquely defined by the action of generators, which can be arbitrary.

Further, assume that B is a $\mathbb{Z}_-$-graded associative dg-algebra which is quasifree, i.e., such that $B_\# \simeq F(E^\bullet)$ is free. Then, the graded scheme $\mathrm{Act}(B,V)_\#$ is, by the above, identified with $|\operatorname{Hom}_{\mathbb{K}}(E\otimes V,V)|$.

(3.3.6) Proposition. *If $f:B_1\to B_2$ is a quasiisomorphism of quasifree associative $\mathbb{Z}_-$-graded dg-algebras, then $f^*:\mathrm{Act}(B_2,V)\to\mathrm{Act}(B_1,V)$ is a quasiisomorphism of dg-schemes.*

We will prove this proposition a little later. Assuming it is true, we give the following definition.

(3.3.7) Definition. *We define* $\mathrm{RAct}(A,V)=\mathrm{Act}(B,V)$*, where $B\to A$ is any quasifree resolution.*

(3.4) Reminder on A_∞-structures. In what follows it will be convenient to use the language of A_∞-structures. This concept goes back to J. Stasheff [St1] for dg-algebras, but here we need the companion concepts for modules (introduced by M. Markl [Ma]) and for morphisms of modules.

(3.4.1) Definition. *Let A be an associative dg-algebra. A left A_∞-module over A is a graded vector space M together with $\mathbb{K}$-multilinear maps*

$$\mu_n : A^{\otimes n} \otimes M \to M, \quad n \geq 0, \quad \deg(\mu_n) = 1 - n,$$

satisfying the conditions:

$$\sum_{i=1}^{n} (-1)^{\bar{a}_1 + \ldots + \bar{a}_{i-1}} \mu_n(a_1, ..., da_i, ..., a_n, m) =$$

$$= \sum_{i=1}^{n-1} (-1)^i \mu_{n-1}(a_1, ..., a_i a_{i+1}, ..., a_n, m) -$$

$$- \sum_{\substack{p,q \geq 0 \\ p+q=n}} (-1)^{q(\bar{a}_1 + \ldots + \bar{a}_p) + p(q-1) + (p-1)q} \mu_p(a_1, ..., a_p, \mu_q(a_{p+1}, ..., a_n, m)).$$

This implies, in particular, that $d_M = \mu_0$ satisfies $d_M^2 = 0$ and μ_1 induces on $H^\bullet_{d_M}(M)$ a structure of a graded left $H^\bullet(A)$-module. A collection of maps μ_n satisfying the conditions of (3.4.1) will be also referred to as an A_∞-action of A on M. An A_∞-action with $\mu_n = 0$ for $n \geq 2$ is the same as a structure of a dg-module in the ordinary sense.

(3.4.2) Definition. *Let A be an associative dg-algebra, M be a left A_∞-module and N be a genuine dg-module over A. An A_∞-morphism $f : M \to N$ is a collection of linear maps*

$$f_n : A^{\otimes n} \otimes M \to N, \quad \deg(f_n) = -n,$$

satisfying the conditions:

$$df_n(a_0, ..., a_n, m) - \sum_{i=1}^{n} (-1)^i f_n(a_1, ..., da_i, ..., a_n, m) =$$

$$\sum_{i=0}^{n-1} (-1)^i f_{n-1}(a_1, ..., a_i a_{i+1}, ..., a_n, m) + \sum_{p=0}^{n} (-1)^{p(n-p)} f_p(a_0, ..., a_p, \mu_p(a_{p+1}, ..., a_n, m)).$$

Again, the conditions imply that $f_0 : M \to N$ is a morphism of complexes and induces a morphism of left $H^\bullet(A)$-modules $H^\bullet(M) \to H^\bullet(N)$.

A_∞-structures have transparent interpretation via bar-resolutions. Let us start with A_∞-modules. Assume that A is $\mathbb{Z}_-$-graded and consider the graded vector space

$$(3.4.3) \qquad \bigoplus_{n=1}^{\infty} A^{\otimes n}[n-1] = \mathrm{Tot}\Big\{ ...A\otimes A\otimes A; \quad A\otimes A; \quad A\Big\},$$

Here Tot means the $\mathbb{Z}_-$-graded vector space associated to a $\mathbb{Z}_-\times\mathbb{Z}_-$-graded one. Let $D(A)$ be the free associative algebra without unit on this graded vector space. The multiplication operation in $D(A)$ will be denoted by $*$. We introduce a differential $d = d' + d''$ on $D(A)$ where d' comes from the tensor product differential on the $A^{\otimes m}$ and d'' is defined on generators by
(3.4.4)

$$d''(a_0\otimes...\otimes a_n) = \sum_{i=0}^{n-1}(-1)^i a_0\otimes...\otimes a_i a_{i+1}\otimes...\otimes a_n - \sum_{i=0}^{n-1}(-1)^i(a_0\otimes...\otimes a_i)*(a_{i+1}\otimes...\otimes a_n)$$

(3.4.5) Proposition. *(a) The differential d satisfies $d^2 = 0$.*
(b) The projection $D(A) \to F(A) \stackrel{m}{\to} A$, where m is the multiplication in A, is a quasiisomorphism.

Proof. Well known: $D(A)$ is the bar-construction of the cobar-construction of A, see [HMS, II§3] for (a) and [HMS, Thm. II.4.4] for (b).

Thus $D(A)$ is a quasifree resolution of A. By comparing (3.4.4) with (3.4.1), we find at once (cf. [Ma]).

(3.4.6) Proposition. *An A_∞-action of A on M is the same as a genuine action (structure of a dg-module) of $D(A)$ on M.*

Similarly, let M be a left A_∞-module over A. We consider the graded vector space

$$(3.4.7) \quad \mathrm{Bar}_A(M) = \bigoplus_{n=1}^{\infty} A^{\otimes n}\otimes M[n-1] = \mathrm{Tot}\Big\{ ... \to A\otimes_{\mathbb{K}} A\otimes_{\mathbb{K}}\otimes M \to A\otimes_{\mathbb{K}} M\Big\},$$

cf. (2.4.4). It has a natural structure of a free left $A_\#$-module. We equip it with the differential

$$(3.4.8) \qquad d(a_0\otimes ...\otimes a_n\otimes m) = \sum_{i=0}^{n}(-1)^{i-1} a_0\otimes ...\otimes da_i\otimes ...\otimes a_n\otimes m +$$

$$+\sum_{i=0}^{n-1}(-1)^i a_0\otimes...\otimes a_i a_{i+1}\otimes...\otimes a_n\otimes m + \sum_{p=0}^{n}(-1)^{p(n-p)} a_0\otimes...\otimes a_p\otimes\mu_{n-p}(a_{p+1}\otimes...\otimes a_n\otimes m).$$

The following is straightforward.

(3.4.9) Proposition. *(a) The differential d satisfies $d^2 = 0$ and makes $\mathrm{Bar}_A(M)$ into a left dg-module over A.*
(b) The projection $\mathrm{Bar}_A(M) \to M$ which on $A^{\otimes n} \otimes M$ is given by μ_n, is a quasi-isomorphism.
(c) Let N be any genuine dg-module over A. Then an A_∞-morphism $M \to N$ is the same as a morphism of dg-modules $\mathrm{Bar}_A(M) \to N$.

(3.5) A model for RAct **classifying A_∞-actions.** Let A be a $\mathbb{Z}_-$-graded associative dg-algebra and V be an ungraded finite-dimensional vector space. We set

$$\widetilde{\mathrm{R}}\,\mathrm{Act}(A, V) = \mathrm{Act}(D(A), V). \tag{3.5.1}$$

So it is a model for $\mathrm{RAct}(A, V)$, defined via the particular quasifree resolution $D(A)$ of A. We postpone till n.(3.6) the discussion of other resolutions and concentrate on this model. By construction, the affine dg-scheme $\widetilde{\mathrm{R}}\,\mathrm{Act}(A, V)$ is the classifier of A_∞-actions. Its coordinate ring $\mathbb{K}[\widetilde{\mathrm{R}}\,\mathrm{Act}(A, V)]$ is the *free* graded commutative algebra on the matrix elements of *indeterminate* linear operators $\mu_n : A^{\otimes n} \otimes V \to V$ while the differential is chosen so as to enforce (3.4.1). In other words, we have:

(3.5.2) Proposition. *For any commutative dg-algebra Λ the set*

$$\mathrm{Hom}_{\text{dg-Alg}}(\mathbb{K}[\widetilde{\mathrm{R}}\,\mathrm{Act}(A, V)], \Lambda)$$

is naturally identified with the set of Λ-(multi)linear A_∞-actions of $A \otimes_{\mathbb{K}} \Lambda$ on $V \otimes_{\mathbb{K}} \Lambda$.

Notice that if A has all its graded components finite-dimensional, then so does $\mathbb{K}[\widetilde{\mathrm{R}}\,\mathrm{Act}(A, V)]$ and therefore $\widetilde{\mathrm{R}}\,\mathrm{Act}(A, V)$ is a dg-manifold.

We now describe a version of the Eilenberg-Moore spectral sequences for the functor $A \mapsto \mathbb{K}[\widetilde{\mathrm{R}}\,\mathrm{Act}(A, V)]$.

(3.5.3) Proposition. *For any $\mathbb{Z}_-$-graded associative dg-algebra A we have natural convergent spectral sequences*

$$\text{(a)} \qquad E_1 = H^\bullet\, \mathbb{K}[\widetilde{\mathrm{R}}\,\mathrm{Act}(A^\bullet_\#, V)] \quad \Longrightarrow \quad H^\bullet\, \mathbb{K}[\widetilde{\mathrm{R}}\,\mathrm{Act}(A, V)];$$

$$\text{(b)} \qquad E_2 = H^\bullet\, \mathbb{K}[\widetilde{\mathrm{R}}\,\mathrm{Act}(H^\bullet(A), V)] \quad \Longrightarrow \quad H^\bullet\, \mathbb{K}[\widetilde{\mathrm{R}}\,\mathrm{Act}(A, V)].$$

Proof. Let us construct the sequence (b), the first one being similar. As a vector space,

$$\mathbb{K}[\widetilde{\mathrm{R}}\,\mathrm{Act}(A, V)] \quad = \quad S\Big(\bigoplus_{n=1}^{\infty} \mathrm{Hom}(V, A^{\otimes n} \otimes V)\Big)$$

and its grading comes from a natural bigrading of which the first component is induced by the grading in A while the other one is the grading in the symmetric algebra induced by the grading of the generators $\deg \operatorname{Hom}(V, A^{\otimes n} \otimes V) = 1 - n$. Similarly, the differential d is a sum $d' + d''$ where d', of bidegree $(1,0)$, is induced by the differential in A and d'', of bidegree $(0,1)$, is induced by the algebra structure in A (i.e., d'' is the differential in $\mathbb{K}[\widetilde{\mathrm{R}}\operatorname{Act}(A^\bullet_\#, V)]$). Thus we have a double complex. Now, since taking cohomology commutes with tensor products over $\mathbb{K}$, we find that

$$H^\bullet_{d'}\, \mathbb{K}[\widetilde{\mathrm{R}}\operatorname{Act}(A, V)] \simeq \mathbb{K}[\widetilde{\mathrm{R}}\operatorname{Act}(H^\bullet(A), V)]$$

as complexes, if we take the differential on the right to be induced by d''. So our statement follows from the standard spectral sequence of a double complex, which converges as the double complex is $\mathbb{Z}_- \times \mathbb{Z}_-$-graded.

(3.5.4) Proposition. *Suppose A is concentrated in degree 0. Then:*
(a) We have $\pi_0 \widetilde{\mathrm{R}}\operatorname{Act}(A, V) = \operatorname{Act}(A, V)$.
(b) For any $\mu \in \operatorname{Act}(A, V)$ the spaces $H^i T^\bullet_\mu \widetilde{\mathrm{R}}\operatorname{Act}(A, V)$ are given by (3.3.4).

Proof. (a) The underlying ordinary scheme of $\widetilde{\mathrm{R}}\operatorname{Act}(A, V)$ is the affine space

$$|\operatorname{Hom}_{\mathbb{K}}(A \otimes V, V)|.$$

The ideal of the subscheme π_0 is the image, under d, of the (-1)st graded component of the coordinate ring. The space of generators of the coordinate ring in degree (-1) is $\operatorname{Hom}_{\mathbb{K}}(V, A \otimes A \otimes V)$ and the ideal in question is exactly given by the associativity conditions (3.3.1).

(b) A direct inspection shows that we have an identification of complexes

$$T^\bullet_\mu \widetilde{\mathrm{R}}\operatorname{Act}(A, V) = \operatorname{Hom}_A(\operatorname{Bar}_A^{\leq -1}(V), V)[1], \tag{3.5.5}$$

so our statement follows from the fact that $\operatorname{Bar}_A(M)$, being a free resolution, can be used to calculate the Ext's.

(3.6) M-homotopies. To prove Proposition 3.3.6, we need a particular nonlinear generalization of the principle (well known in the usual homological algebra) that any two free resolutions of a module are homotopy equivalent. In order for such a statement to be useful, it needs to employ a concept of homotopy which is preserved under functorial constructions on algebras. The usual notion of chain homotopy of morphisms of complexes is preserved only under additive functors and so is not good for our purposes. A better concept of homotopy in the nonlinear context, which we now describe, goes back to Quillen [Q, Ch. 1, Def. 4] cf. also [BG, §6], [Le, Ch. II, §1].

Let A, B be associative dg-algebras over $\mathbb{K}$ and $(f_t : A \to B)_{t\in[0,1]}$ be a smooth family of dg-homomorphisms parametrized by the unit interval in $\mathbb{R}$. Then, for each t, the derivative $f_t' = \frac{d}{dt} f_t$, satisfies

$$f_t'(ab) = f_t(a)f_t'(b) + f_t'(a)f_t(b)$$

i.e., it is a degree 0 derivation $A \to B$ with respect to the A-bimodule structure on B given by f_t. Also, f_t' commutes with the differentials in A and B, i.e., $[d, f_t'] = 0$.

(3.6.1) Definition. *An M-homotopy (M for multiplicative) is a pair $(f_t, s_t)_{t\in[0,1]}$ where (f_t) is as above and $s_t : A \to B[-1]$ is a smooth family of degree (-1) derivations (with respect to the bimodule structures given by the f_t) such that $f_t' = [d, s_t]$.*

(3.6.2) Proposition. *For an M-homotopy, f_0 and f_1 induce the same morphism $H(A) \to H(B)$.*

Proof. Clear, as f_t', being homotopic to 0 in the usual sense of cochain complexes, induces 0 on the homology.

(3.6.3) Remark. A polynomial M-homotopy is the same as a morphism of dg-algebras

$$A \to B \otimes_{\mathbb{K}} \Big(\mathbb{K}[t, \epsilon], \;\; \deg t = 0, \;\; \deg(\epsilon) = +1, \;\; dt = \epsilon \Big).$$

The dg-algebra $\mathbb{K}[t, \epsilon]$ on the right is $\mathbb{Z}_+$-graded, so it is formally outside the framework of this paper. Nevertheless, it is quasiisomorphic to $\mathbb{K}$, so from a wider derived-categorical point of view an M-homotopy should be thought of as representing one morphism $A \to B$.

The following construction was presented by M. Kontsevich in his course on deformation theory (Berkeley 1994).

(3.6.4) Proposition. *Let B, C be $\mathbb{Z}_-$-graded dg-algebras such that B is quasifree and C is acyclic in degrees < 0. Let $f_0, f_1 : B \to C$ be two morphisms of dg-algebras inducing the same morphism $H^0(B) \to H^0(C)$. Then there exists a polynomial M-homotopy between f_0 and f_1.*

Proof. As B is quasifree, let us write $B_\# = F(E^\bullet)$, for some $\mathbb{Z}_-$-graded vector space $E^\bullet$ of generators. A morphism $g : B \to C$ is uniquely defined by its restriction on the generators which furnishes a family of linear maps $g^{(i)} : E^{-i} \to C^{-i}$. Conversely, any choice of such maps which is compatible with the differentials, defines a homomorhism. Similarly, a derivation $\sigma : B \to C$ (with respect to the bimodule structure on C given by g), is uniquely described by its restriction on generators which gives linear maps $\sigma^{(i)} : E^{-i} \to C^{-i-1}$.

We now construct a family of homomorphisms $(f_t) : B \to C$, interpolating between f_0, f_1 inductively, by constructing successively the $f_t^{(i)}$, $i = 0, 1, ...$ To start, we define $f_t^{(0)}$ by linear interpolation: $f_t^{(0)}(e) = (1-t)f_0(e) + tf_1(e)$, $e \in E^0$. On this stage the compatibility with the differential does not yet arise. By construction, the images of $f_t^{(0)}(e)$ in $H^0(C)$ are independent on t and therefore $(d/dt)f_t^{(0)}(e)$ takes values in $\mathrm{Im}\{d : C^{-1} \to C^0\}$. So we can find a polynomial family of maps $s_t^{(0)} : E_1^0 \to C^{-1}$ such that $(d/dt)f_t^{(0)}(e) = ds_t^{(0)}(e)$. To continue, we need to define $f_t^{(1)} : E^{-1} \to C^{-1}$ in such a way that

$$df_t^{(1)}(e) = f_t^{(0)}(de) \tag{3.6.5}$$

But $f_t^{(0)}(de)$, being a linear interpolation between $f_0(de) = df_0(e)$ and $f_1(de) = df_1(e)$, lies, for any t, in the image of d. Therefore we can choose a polynomial family $(f_t^{(1)})$ interpolating between $f_0^{(1)}$ and $f_1^{(1)}$ and satisfying (3.6.5). Next, for $e \in E^{-1}$, we have

$$d\left(\frac{d}{dt}f_t^{(1)}(e) - s_t^{(0}(de)\right) = d\left(\frac{d}{dt}f_t^{(0)}(de)\right) - d(s_t^{(0)}(de)) = d(s_t^{(0)}(de)) - d(s_t^{(0)}(de)) = 0$$

and because B_2 is acyclic in degrees ≤ -1, we can find a polynomial family of linear maps $(s_t^{(1)} : E_1^{-1} \to B_2^{-2})$ such that

$$d(s_t^{(1)}(e)) = \frac{d}{dt}f_t^{(1)}(e) - s_t^{(0)}(de)$$

which is the first in the series of conditions defining an M-homotopy. We then continue in this way, defining successively the $f_t^{(i)}$ and $s_t^{(i)}$ on E^{-i} and extending them to homomorphisms (resp. derivations) on the subalgebra generated by $E^{-i}, ..., E^0$. This furnishes a required M-homotopy.

Let now $A = F(E^\bullet)$ be the free associative algebra on the $\mathbb{Z}_-$-graded vector space $E^\bullet$ (no differential). Then the quasiisomorphism $\alpha : D(A) \to A$ described in (3.4.5)(b), has a natural right inverse $\beta : A \to D(A)$, so that $\alpha\beta = \mathrm{Id}_A$. More precisely, β is defined on the space of generators $E \subset A$ to identify it with the natural copy of E inside $A \subset F(A) \subset D(A)$ and then extended to the entire A because A is free. Thus β is also a quasiisomorphism.

(3.6.6) Proposition. *The composition $\beta\alpha : D(A) \to D(A)$ is M-homotopic to the identity of $D(A)$.*

Proof. If $E^\bullet$ is in degree 0, then so is A and thus we can apply Proposition 3.6.4 to $B = C = D(A)$. If $E^\bullet$ is not concentrated in degree 0, then we notice that $D(A)$ in fact comes from a $\mathbb{Z}_- \times \mathbb{Z}_-$-graded dg-algebra and that we can mimic all the steps in the proof of (3.6.4), using the induction in the second component of the bidegree.

(3.7) Proof of Proposition 3.3.6. Proposition 3.6.6 implies the following.

(3.7.1) Corollary. *If $A = F(E^\bullet)$ is free with trivial differential, then $\widetilde{\mathrm{R}}\,\mathrm{Act}(A,V)$ is quasiisomorphic to $\mathrm{Act}(A,V) = |\,\mathrm{Hom}_{\mathbb{K}}(E^\bullet \otimes V, V)|$.*

Proof. As we pointed out before, the correspondence $A \mapsto \mathrm{Act}(A,V)$ is contravariantly functorial in A; equivalently, $\mathbb{K}[\mathrm{Act}(A,V)]$ depends on A in a covariant way. Thus the maps α, β between A and $D(A)$ induce morphisms of commutative dg-algebras α_*, β_* from $\mathbb{K}[\mathrm{Act}(A,V)]$ to $\mathbb{K}[\mathrm{Act}(D(A),V)] = \mathbb{K}[\widetilde{\mathrm{R}}\,\mathrm{Act}(A,V)]$ and back, with $\alpha_*\beta_* = \mathrm{Id}$. Further, the polynomial M-homotopy between $\beta\alpha$ and Id, constructed in (3.6.6), is also inherited, because of Remark 3.6.3, in functorial constructions such as passing to $\mathbb{K}[\mathrm{Act}(-,V)]$. This proves the statement.

(3.7.2) Proposition. *If $B \xrightarrow{p} A$ is any quasifree associative dg-resolution, then $\mathbb{K}[\mathrm{Act}(B,V)]$ is naturally quasiisomorphic to $\mathbb{K}[\widetilde{\mathrm{R}}\,\mathrm{Act}(A,V)]$ and therefore it is independent, up to a quasiisomorphism, of the choice of B.*

Proof. By Corollary 3.6.5, $\mathbb{K}[\mathrm{Act}(B_\#,V)]$ is quasiisomorphic to $\mathbb{K}[\widetilde{\mathrm{R}}\,\mathrm{Act}(B_\#,V)]$, the quasiisomorphism being induced by the map α. Proposition 3.5.3(a) implies then that the map

$$\mathbb{K}[\mathrm{Act}(B,V)] \to \mathbb{K}[\widetilde{\mathrm{R}}\,\mathrm{Act}(B,V)]$$

is also a quasiisomorphism as it induces an isomorphism of the first terms of the spectral sequences described in 3.5.3(a). Further, the spectral sequence (3.5.3)(b) shows that the morphism

$$p_* : \mathbb{K}[\widetilde{\mathrm{R}}\,\mathrm{Act}(B,V)] \to \mathbb{K}[\widetilde{\mathrm{R}}\,\mathrm{Act}(A,V)]$$

is a quasiisomorphism. This proves our statement.

Thus we have established Proposition 3.3.6.

(3.8) The derived linearity locus. Let S be a $\mathbb{Z}_-$-graded dg-scheme and A be a $\mathbb{Z}_-$-graded associative dg-algebra. Let M, N be two quasicoherent dg-sheaves on S such that $M_\#, N_\#$ are locally free over $\mathcal{O}_{S^\#}$. We assume that the generators of $M_\#$ are in degrees ≤ 0 and those of $N_\#$ are in degrees ≥ 0. Suppose that M, N are made into dg-modules over $A \otimes_{\mathbb{K}} \mathcal{O}_S$ and we have a morphism $f : M \to N$ of $\mathcal{O}_S$-dg-modules (but not necessarily of $A \otimes_{\mathbb{K}} \mathcal{O}_S$-dg-modules). According to the general approach of homological algebra, we define the derived linearity locus $\mathrm{RLin}_A(f)$ as the derived fiber product (2.8)

$$\begin{array}{ccc} \mathrm{RLin}_A(f) & \longrightarrow & S \\ \big\downarrow & & \big\downarrow f \\ |\,\mathrm{R}\,\mathcal{H}om_{A\otimes\mathcal{O}_S}(M,N)| & \xrightarrow{|\rho|} & |\mathcal{H}om_{\mathcal{O}_S}(M,N)|. \end{array} \tag{3.8.1}$$

Here $\mathrm{R}\,\mathcal{H}om_{A\otimes\mathcal{O}_S}(M,N) = \mathcal{H}om_{A\otimes\mathcal{O}_S}(P,N)$, where $P\to M$ is a resolution by a dg-module such that:

(1) P is $\mathbb{Z}_-$-graded and $P_\#$ is locally (on the Zariski topology of S) projective over $A_\#\otimes\mathcal{O}_{S\#}$.

If, in addition, we have a stronger condition, namely:

(2) The morphism ρ is a termwise surjective morphism of cochain complexes.

then the derived fiber product coincides with the usual fiber product, as the morphism $|\rho|$ is flat.

One example of a resolution satisfying (1) and (2) above is the bar-resolution $\mathrm{Bar}_A(M)\to M$, see (3.4.7) (we consider M as an A_∞-module with $\mu_i = 0, i\geq 2$). The following is then a standard application of the Eilenberg-Moore spectral sequences for the derived Hom and tensor product of dg-modules.

(3.8.2) Proposition. *The derived linearity locus is independent, up to a quasi-isomorphism, on the choice of P satisfying (1) and (2).*

We denote by

$$\widetilde{\mathrm{R}}\,\mathrm{Lin}_A(f) = |\mathcal{H}om^\bullet_{A\otimes\mathcal{O}_S}(\mathrm{Bar}_A(M),N)| \times_{|\mathcal{H}om_{\mathcal{O}_S}(M,N)|} S \tag{3.8.3}$$

the particular model for RLin obtained by using the bar-resolution. (we consider M as an A_∞-module with $\mu_i=0, i\geq 2$). Let us note some additional properties of this model. First, it can be applied in a more general situation. Namely, let A and N be as before, but assume that M is only an A_∞-module over A. In this case, as $\mathrm{Bar}_A(M)$ makes sense, we define $\widetilde{\mathrm{R}}\,\mathrm{Lin}_A(f)$ by (3.8.3). Notice that while we can view an A_∞-morphism $f: M\to N$ as a morphism of $D(A)$-dg-modules, (still denoted f), the A_∞-version $\widetilde{\mathrm{R}}\,\mathrm{Lin}_A(f)$ is much more economical in size than any of the models for $\mathrm{RLin}_{D(A)}(f)$ given by (3.8.1-2), especially than $\widetilde{\mathrm{R}}\,\mathrm{Lin}_{D(A)}(f)$.

(3.8.4) Proposition. $\widetilde{\mathrm{R}}\,\mathrm{Lin}_A(f)$ *is the complex of vector bundles on S*

$$\mathrm{Cone}\left\{\mathcal{O}_S \xrightarrow{\delta f} \mathcal{H}om_{\mathcal{O}_S}(\mathrm{Bar}_A(M),N)\right\}[1]$$

considered as a dg-scheme. Here $\delta f\in\mathcal{H}om_{\mathcal{O}_S}(A\otimes M,N)$ takes $a\otimes m\mapsto f(a\otimes m) - af(m)$.

As with $\widetilde{\mathrm{R}}\,\mathrm{Act}$, the construction of $\widetilde{\mathrm{R}}\,\mathrm{Lin}$ can be interpreted via A_∞-structures.

(3.8.5) Proposition. *(a) The natural morphism $p:\widetilde{\mathrm{R}}\,\mathrm{Lin}_A(f)\to S$ is smooth and the induced morphism $p^*f: p^*M\to p^*N$ is an A_∞-morphism of dg-modules over $A\otimes\mathcal{O}_{\widetilde{\mathrm{R}}\,\mathrm{Lin}_A(f)}$.*

(b) For any commutative dg-algebra Λ the set $\mathrm{Hom}_{\text{dg-Sch}}(\mathrm{Spec}(\Lambda), \widetilde{\mathrm{R}}\,\mathrm{Lin}_A(f))$ *is identified with the set of data $(g, h_1, h_2, ...)$ where $g : \mathrm{Spec}(\Lambda) \to S$ is a morphism of dg-schemes and $h_n : A^{\otimes n} \otimes_{\mathbb{K}} g^*M \to g^*N$ are such that $(g^*f, h_1, h_2, ...)$ is an A_∞-morphism $g^*M \to g^*N$.*

Informally, $\widetilde{\mathrm{R}}\,\mathrm{Lin}_A(f)$ is obtained by adding to $\mathcal{O}_S$ new free generators which are matrix elements of interdeterminate higher homotopies $h_i : A^{\otimes i} \otimes M \to N$, $i \geq 1$ and arranging the differential there so as to satisfy (3.4.2).

(3.9) The derived A-Grassmannian. We place ourselves in the situation of the beginning of this section, So A is a finite-dimensional $\mathbb{K}$-algebra and M a finite-dimensional A-module. By applying the construction of the derived space of actions to any fiber of the tautological bundle $\widetilde{V}$ on $G(k, M)$, we get a dg-scheme $\mathrm{RAct}(A, \widetilde{V}) \xrightarrow{q} G(k, M)$. If we take a quasifree resolution $B \to A$ with finitely many generators in each degree, then $\mathrm{RAct}(A, \widetilde{V})$ will be a dg-manifold. For example, the model $\widetilde{\mathrm{R}}\,\mathrm{Act}(A, \widetilde{V})$ obtained via the bar-resolution $D(A)$, satisfies this property. Thus $q^*\widetilde{V}$ is a dg-module over B.

(3.9.1) Definition. *The derived A-Grassmannian $RG_A(k, M)$ is defined as the derived linearity locus* $\mathrm{RLin}_B(f)$*, where $f : q^*\widetilde{V} \to \underline{M}$ is the tautological morphism from (3.2.3).*

A smaller model can be obtained by taking $B = D(A)$, viewing a dg-module over $D(A)$ as an A_∞-module over A and apply the construction of the derived linearity locus for A_∞-modules described in (3.8). This model is a dg-manifold.

(3.9.2) Theorem. *(a) We have $\pi_0\, RG_A(k, M) = G_A(k, M)$.*
(b) For any A-submodule $V \subset M$ with $\dim_{\mathbb{K}}(M) = k$*, we have*

$$H^i T^\bullet_{[V]} RG_A(k, M) = \mathrm{Ext}^i_A(V, M/V).$$

Proof. (a) follows from similar properties of RAct, RLin (Propositions 3.5.4 and 3.8.4). To see (b), notice that we have an identification in the derived category:

$$\mathrm{RHom}_A(V, M/V) \sim \mathrm{Cone}\Big\{\mathrm{RHom}_A(V, V) \to \mathrm{RHom}_A(V, M)\Big\}[1].$$

To be specific, we will consider the model for RG_A obtained by using $\widetilde{\mathrm{R}}\,\mathrm{Act}$ and the A_∞-version of $\widetilde{\mathrm{R}}\,\mathrm{Lin}$. Then, denoting $\mu : A \otimes V \to V$ the induced A-action on the submodule V, we have, by Proposition 3.8.4, an identification of complexes

$$T^\bullet_{[V]} RG_A(k, M) = \mathrm{Cone}\Big\{T^\bullet_{([V],\mu)} \widetilde{\mathrm{R}}\,\mathrm{Act}(A, \widetilde{V}) \to T^\bullet_{([V],\mu)} |\mathcal{H}om_{\mathcal{O}_{\widetilde{\mathrm{R}}\,\mathrm{Act}(A, \widetilde{V})}}(\mathrm{Bar}_A(q^*\widetilde{V}), \underline{M})|\Big\}.$$

The dg-scheme $\widetilde{\mathrm{R}}\,\mathrm{Act}(A,\widetilde{V})$ is a fibration over $G(k,M)$, and $|\mathcal{H}om_{\mathcal{O}_{\widetilde{\mathrm{R}}\,\mathrm{Act}(A,\widetilde{V})}}(\mathrm{Bar}_A(q^*\widetilde{V}),\underline{M})|$ is a fibration over $\widetilde{\mathrm{R}}\,\mathrm{Act}(A,\widetilde{V})$ so it also a fibration over $G(k,M)$. The tangent bundle of each of these fibrations fits into short exact sequence involving the relative tangent bundle and the pullback of $TG(k,M)$. Let us write the corresponding exact sequences for fibers of the tangent bundles. Using Propositions 3.5.4 and 3.8.4, we can write them as follows:

$$0 \to \mathrm{Hom}_A(\mathrm{Bar}_A^{\leq -1}(V),V) \to T^\bullet_{([V],\mu)}\widetilde{\mathrm{R}}\,\mathrm{Act}(A,\widetilde{V}) \to T_{[V]}G(k,M) \to 0,$$

$$0 \to \mathrm{Hom}_A(\mathrm{Bar}_A(V),M) \to T^\bullet_{([V],\mu)}|\mathcal{H}om_{\mathcal{O}_{\widetilde{\mathrm{R}}\,\mathrm{Act}(A,\widetilde{V})}}(\mathrm{Bar}_A(q^*\widetilde{V}),\underline{M})| \to T_{[V]}G(k,M) \to 0.$$

This means that in the cone the two copies of $T_{[V]}G(k,M) \to 0$ will cancel out, up to quasiisomorphism, and we conclude that

$$T_{[V]}RG_A(k,M) = \mathrm{Cone}\Big\{\mathrm{Hom}_A(\mathrm{Bar}_A^{\leq -1}(V),V) \to \mathrm{Hom}_A(\mathrm{Bar}_A(V),M)\Big\}[1],$$

whence the statement.

(3.9.3) Remarks. (a) Instead of working with the module structures on the fibers of the universal subbundle $\tilde{V}$ on $G(k,M)$, we could equally well work with module structures on the fibers of the universal quotient bundle $M/\tilde{V}$ and modify the approach of (3.2) accordingly.

(b) Taking $M = A$, set $\mathcal{J}(k,A) = G_A(k,A)$ (the scheme of ideals in A of dimension k). When A is commutative, there is a derived analog $R\mathcal{J}(k,A)$ of $\mathcal{J}(k,A)$ different from $RG_A(k,A)$ and whose construction will be described in detail in [CK]. The approach is based on realizing $\mathcal{J}(k,A)$ via two constructions similar but not identical to those described in (3.2). The first one is the space $\mathcal{C}(W) \subset \mathrm{Hom}(S^2W,W)$ of all commutative algebra structures on a finite-dimensional vector space W. Applying this to fibers of the bundle $A/\tilde{V}$ on $G(k,A)$, we get a fibration $q : \mathcal{C}(A/\tilde{V}) \to G(k,V)$ and a vector bundle morphism $g : \underline{A} \to q^*(A/\tilde{V})$ on $\mathcal{C}(A/\tilde{V})$. Fibers of both these bundles are commutative algebras and $\mathcal{J}(k,A)$ is the *homomorphicity locus* of g, i.e., the subscheme of points of the base such that the corresponding morphism of the fibers is an algebra homomorphism. The dg-manifold $R\mathcal{J}(k,A)$ is obtained by taking the derived versions of these steps. It will be used in constructing the derived Hilbert scheme mentioned in (0.4).

4. Derived Quot schemes

In this section we will apply the construction of $RG_A(k,M)$ of §3 to the case of interest in geometry, when $A=\bigoplus_{i\geq 0} H^0(X,\mathcal{O}_X(i))$ for a projective scheme X and $M=\bigoplus_{i=p}^q H^0(X,\mathcal{F}(i))$ for a coherent sheaf $\mathcal{F}$ on X. In this situation all objects acquire extra grading and to avoid confusion, we sharpen our terminology.

(4.1) Conventions on grading. We will consider bigraded vector spaces $V = V_\bullet^\bullet = \bigoplus_{p,q} V_q^p$. The lower grading will be called *projective* and the upper one, *cohomological*. By a bigraded complex we mean a bigraded vector space with a differential d having degree 1 in the upper grading and 0 in the lower one. Tensor products $V_\bullet^\bullet \otimes W_\bullet^\bullet$ of bigraded complexes are defined in the usual way and the symmetry map $V_\bullet^\bullet \otimes W_\bullet^\bullet \to W_\bullet^\bullet \otimes V_\bullet^\bullet$ is defined by the Koszul sign rule involving only the upper grading. The concepts of a bigraded (commutative) dg-algebra, bigraded A_∞- algebra etc. will be understood accordingly, with only the upper grading contributing to the sign factors.

Given a (lower) graded associative algebra $A=\bigoplus A_i$ and its left graded modules $M=\bigoplus M_i$, $N=\bigoplus N_i$ we define the $\mathrm{Ext}_A^{i,0}(M,N)$ to be the derived functors of $\mathrm{Hom}_A^0(M,N)$, i.e., of the Hom functor in the category of graded modules.

(4.2) Derived A-Grassmannian in the graded case. Let $A=\bigoplus_{i\geq 0} A_i$ be a graded associative algebra with $A_0=\mathbb{K}$ and $\dim(A_i)<\infty$ for all i. Let $M=\bigoplus M_i$ be a finite-dimensional graded A-module; thus there are $p\leq q$ such that $M_i=0$ unless $i\in[p,q]$. If $k=(k_i)$ is a sequence of nonnegative integers, we have introduced in (1.3) the graded A-Grassmannian $G_A(k,M)$. The construction of the derived A-Grassmannian $RG_A(k,M)$ in Section 3, can be repeated without changes in the graded case, if we consider everywhere only morphisms of degree 0 with respect to the lower grading and replace the functor Hom with Hom^0.

(4.2.1) Theorem. *Up to quasiisomorphism, the graded version of the derived A-Grassmannian is a dg-manifold satisfying the conditions:*

$$\pi_0 RG_A(k,M)=G_A(k,M),\quad H^i T_{[V]} RG_A(k,M)=\mathrm{Ext}_A^{i,0}(V,M/V).$$

Proof. The only issue that needs to be addressed is the existence of a model for $RG_A(k,M)$ which is of finite type, as A is now infinite-dimensional over $\mathbb{K}$. More precisely, we need to show that the particular model $\widetilde{\mathrm{R}}\mathrm{Act}(A,V)$ and $\widetilde{\mathrm{R}}\mathrm{Lin}_A(f)$ are of finite type in the bigraded context as well (then they will be dg-manifolds by construction).

To see this, recall that $\widetilde{\mathrm{R}}\mathrm{Act}(A,V)$ is the affine dg-scheme whose coordinate algebra $\mathbb{K}[\widetilde{\mathrm{R}}\mathrm{Act}(A,V)]$ is the free (upper) graded commutative algebra on the matrix elements of indeterminate linear maps $A^{\otimes n}\otimes V\to V$ *of degree 0 with respect to the lower grading.* Since V (a graded subspace of M) is concentrated in only finitely many degrees (from p to q), and since we can disregard $A_0 = \mathbb{K}$, there are only finitely many possibilities for nonzero maps $A_{i_1}\otimes...\otimes A_{i_n}\otimes V_j \to V_{i_1+...+i_n+j}$, $i_\nu > 0$, $p \le j \le q$. Each such possibility gives a finite-dimensional space of maps. This implies that each (upper) graded component of $\mathbb{K}[\widetilde{\mathrm{R}}\mathrm{Act}(A, V)]$ is finite-dimensional, so $\widetilde{\mathrm{R}}\mathrm{Act}(A, V)$ is a dg-manifold. Proposition 3.5.4 now holds in the graded context, with Hom and Ext replaced everywhere by Hom^0 and Ext^0.

Further, if we use the same convention in (3.8), we get that the graded version $\widetilde{\mathrm{R}}\mathrm{Lin}$ is also a dg-manifold. The theorem is proved.

(4.3) The derived Quot scheme. Let now $X \subset \mathbb{P}^n$ be a *smooth* projective variety, $\mathcal{F}$ a coherent sheaf on X and $h \in \mathbb{Q}[t]$ a polynomial. Let A be the graded coordinate algebra of X and M the graded A-module corresponding to $\mathcal{F}$, see (1.2).

(4.3.1) Definition. *The derived Quot scheme is defined as*

$$RSub_h(\mathcal{F}) := RG_A(h, M_{[p,q]}) \quad \textit{for } 0 \ll p \ll q.$$

Here $RG_A(h, M_{[p,q]})$ is the graded version of the derived Grassmannian constructed in (4.2).

The well-definedness of $RSub_h(\mathcal{F})$ up to isomorphism in the derived category $\mathcal{D}Sch$ is part (a) of the following theorem which is the main result of this paper.

(4.3.2) Theorem. *(a) For $0 \ll p \ll p' \ll q' \ll q$ the natural projection*

$$RG_A(h, M_{[p,q]}) \to RG_A(h, M_{[p',q']})$$

is a quasiisomorphism of dg-manifolds.

(b) $\pi_0 RSub_h(\mathcal{F}) = Sub_h(\mathcal{F})$.

(c) If $\mathcal{K} \subset \mathcal{F}$ has Hilbert polynomial h, then

$$H^i T^\bullet_{[\mathcal{K}]} RSub_h(\mathcal{F}) \simeq \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{K}, \mathcal{F}/\mathcal{K}), \quad i \ge 0.$$

Proof. Part (b) follows from Theorem 1.4.1 and (4.2.1). Part (a) would follow from (b)(c) in virtue of the "Whitehead theorem" (2.5.9). More precisely, we need to apply (c) to the dg-scheme $RSub_h(\mathcal{F}) \otimes \mathbb{F}$ for any field extension $\mathbb{F}$ of $\mathbb{K}$. This scheme is just the $RSub$ scheme corresponding to the sheaf $\mathcal{F} \otimes \mathbb{F}$ on the scheme $X \otimes \mathbb{F}$. So we concentrate on (c) and start with the following.

(4.3.3) Proposition. *(a) If $\mathcal{F}, \mathcal{G}$ are coherent sheaves on X with corresponding graded A-modules $M = Mod(\mathcal{F})$ and $N = Mod(\mathcal{G})$ respectively, then*

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \varinjlim_{p} \mathrm{Ext}^{i,0}_A(M_{\ge p}, N_{\ge p}),$$

and the limit is achieved.

(b) There exists an integer p such that

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{K}, \mathcal{F}/\mathcal{K}) = \mathrm{Ext}^{i,0}_A(W_{\geq p}, M_{\geq p}/W_{\geq p}), \quad W = \mathrm{Mod}(\mathcal{K}),$$

for all subsheaves $\mathcal{K}$ of $\mathcal{F}$ representing $\mathbb{K}$-points of $Sub_h(\mathcal{F})$, where W is the graded A-module corresponding to $\mathcal{K}$ and M is the graded A-module corresponding to $\mathcal{F}$.

Proof. Part (a) follows from Serre's theorem (1.2.2); part (b) follows from (a), from semicontinuity of the rank of a matrix and from the fact that $Sub_h(\mathcal{F})$ is a scheme of finite type.

We now continue the proof of (4.3.2)(c). Since X is smooth, $\mathrm{Ext}^{i,0}_A \neq 0$ for only finitely many i's. In view of Proposition 4.3.3 we are reduced to

(4.3.4) Proposition. *Let M, N, be any finitely generated graded A-modules. Then for any fixed i there exists an integer q_0 such that*

$$\mathrm{Ext}^{i,0}_A(M, N) = \mathrm{Ext}^{i,0}_A(M_{\leq q}, N_{\leq q})$$

for all $q \geq q_0$. Moreover, if M_s and N_s vary in a family parametrized by a projective scheme S, then q_0 can be chosen independent on $s \in S$.

Proof. Assume first that M is free, i.e. $M = A \otimes_{\mathbb{C}} E_\bullet$, with $E_\bullet$ a finite dimensional graded $\mathbb{K}$-vector space. If $i = 0$, we have obviously

$$\mathrm{Hom}^0_A(M_{\leq q}, N_{\leq q}) = \mathrm{Hom}^0_A(M, N) = \mathrm{Hom}^0_{\mathbb{K}}(E_\bullet, N) \tag{4.3.5}$$

whenever q exceeds the maximum of the degrees of the nonzero graded components of $E_\bullet$.

Next we claim that (for M free)

$$\mathrm{Ext}^{i,0}_A(M_{\leq q}, N_{\leq q}) = 0, \text{ for all } i > 0, \text{ all } q \geq 0 \text{ and any } N. \tag{4.3.6}$$

Indeed, the long exact sequence

$$0 \longrightarrow M_{\geq q+1} \longrightarrow M \longrightarrow M_{\leq q} \longrightarrow 0$$

induces (for $i > 0$) an exact sequence

$$\mathrm{Ext}^{i-1,0}_A(M_{\geq q+1}, N_{\leq q}) \longrightarrow \mathrm{Ext}^{i,0}_A(M_{\leq q}, N_{\leq q}) \longrightarrow \mathrm{Ext}^{i,0}_A(M, N_{\leq q}).$$

But the last term in the above sequence vanishes, since M is free. Hence (4.3.6) follows from:

(4.3.7) Lemma. *For any graded A-modules M and N and all $i \geq 0$, we have* $\mathrm{Ext}_A^{i,0}(M_{\geq q+1}, N_{\leq q}) = 0$.

Proof of Lemma (4.3.7). For $i = 0$ the statement is obvious, as $M_{\geq q+1}$ and $N_{\leq q}$ have nontrivial graded components only in disjoint ranges of degrees. For $i > 0$, the groups $\mathrm{Ext}_A^{i,0}(M_{\geq q+1}, N_{\leq q})$ are calculated as the cohomology of the complex $\mathrm{Hom}_A^0(P^\bullet, N_{\leq q})$, where $P^\bullet$ is a free homogeneous resolution of $M_{\geq q+1}$. This resolution can be chosen such that each P^j is concentrated in degrees (with respect to the lower grading) at least $q+1$, so $\mathrm{Hom}_A^0(P^j, N_{\leq q}) = 0$ for all j and the lemma follows.

Proposition 4.3.4 is therefore true for M free. If M is now arbitrary, let

$$F^\bullet = \{\cdots \longrightarrow F^{-1} \longrightarrow F^0\} \longrightarrow M$$

be a free resolution with $F^{-j} = A \otimes_{\mathbb{K}} E_\bullet^{-j}$ and each $E_\bullet^{-j}$ a finite dimensional graded (by lower grading) vector space. The truncation $F^\bullet_{\leq q}$ is then a resolution of $M_{\leq q}$, but it is not free anymore. However, by (4.3.6) and the "abstract De Rham theorem" $\mathrm{Ext}_A^{i,0}(M_{\leq q}, N_{\leq q})$ is still calculated by the cohomology of the complex $\mathrm{Hom}_A^0(F^\bullet_{\leq q}, N_{\leq q})$. By (4.3.5), for some fixed i, the first $i+1$ terms of this complex will be the same as the first $i+1$ terms of $\mathrm{Hom}_A^0(F^\bullet, N)$ whenever $q \geq$ maximum of the degrees of the nonzero graded components of all E^{-j}, $0 \leq j \leq i+1$. Hence

$$\mathrm{Ext}_A^{j,0}(M, N) = \mathrm{Ext}_A^{j,0}(M_{\leq q}, N_{\leq q})$$

for all $j \leq i$. Finally, notice that the above proof also shows the existence of a lower bound for q_0 in a family of modules parametrized by any projective scheme S. Such a family of modules is just a graded $A \otimes_{\mathbb{K}} \mathcal{O}_S$-module $\mathcal{M}$ with graded components being locally free of finite rank over $\mathcal{O}_S$. Because S is projective, we can find a resolution of $\mathcal{M}$ by $A \otimes \mathcal{O}_S$-modules of the form $\mathcal{F}^j = A \otimes \mathcal{E}^j_\bullet$ where $\mathcal{E}^j_\bullet$ is a graded vector bundle on S such that $\bigoplus_p \mathcal{E}^i_p$ has finite rank, and then the above arguments apply word by word.

This concludes the proof of Proposition 4.3.4 and of Theorem 4.3.2.

(4.3.8) Remark. If an algebraic group G acts on $\mathcal{F}$ by automorphisms, then we have an induced action on $Sub_h(\mathcal{F})$. The above construction of $RSub_h(\mathcal{F})$ via the derived A-Grassmannian and the model for the latter via the bar-resolution (4.2.1) immediately imply that G acts on $RSub_h(\mathcal{F})$ by automorphisms of dg-manifolds. A case particularly important for constructing the derived moduli stack of vector bundles on X is $\mathcal{F} = \mathcal{O}_X(-N)^{\oplus r}$, $N, r \gg 0$ and $G = GL_r$. The G-action on an appropriate open part of $RSub$ gives rise to a groupoid in the category of dg-manifolds, and such groupoids provide, as $N, r \to \infty$, more and more representative charts for the moduli (dg-)stack. The exact way of gluing such charts (by quasiisomorphisms) into a global dg-stack requires a separate treatment.

(4.4) Independence of $RSub(\mathcal{F})$ on the projective embedding. Clearly, the concept of the Hilbert polynomial of a coherent sheaf on X depends on the choice of a very ample line bundle L (the pullback of $\mathcal{O}(1)$ under the projective embedding). Accordingly, the scheme $Sub_h(\mathcal{F})$, see (1.1.1) depends on the choice of L. To emphasize this dependence, let us denote it $Sub_h^L(\mathcal{F})$. It is well known, however, that the union

$$Sub(\mathcal{F}) = \coprod_{h\in\mathbb{Q}[t]} Sub_h^L(\mathcal{F})$$

depends on X and $\mathcal{F}$ but not on L, as it parametrizes subsheaves in $\mathcal{F}$ with flat quotients. The analog of this classical statement is the following fact.

(4.4.1) Theorem. *Let L_1, L_2 be two very ample line bundles on X. Then we have an isomorphism*

$$\coprod_{h\in\mathbb{Q}[t]} RSub_h^{L_1}(\mathcal{F}) \quad \sim \quad \coprod_{k\in\mathbb{Q}[t]} RSub_k^{L_2}(\mathcal{F})$$

in the derived category of dg-schemes (of infinite type).

Thus we have a well defined, up to quasiisomorphism, dg-scheme which we can denote $RSub(\mathcal{F})$.

Proof. We begin with the following simple general fact. Let X be a dg-manifold and let Z be a connected component of $\pi_0(X)$. Choose an open subscheme $Y^0 \subset X^0$ such that $Y^0 \cap \pi_0(X) = Z$ and set $\mathcal{O}_Y^\bullet = \mathcal{O}_X^\bullet\ |_{Y^0}$. Then Y is a dg-manifold. Moreover, if we have Y_1 and Y_2 as above, then they are quasiisomorphic to $Y_1 \cap Y_2$. The discussion above implies the following.

(4.4.2) Proposition. *X is quasiisomorphic to a dg-manifold which is a disjoint union of open submanifolds, each of them containing exactly one connected component of $\pi_0(X)$.*

To continue the proof of (4.4.1), put

$$A := \bigoplus_{m\geq 0} H^0(X, L_1^{\otimes m}), \quad B := \bigoplus_{n\geq 0} H^0(X, L_2^{\otimes n}), \quad C := \bigoplus_{m,n\geq 0} H^0(X, L_1^{\otimes m}\otimes L_2^{\otimes n}),$$

and

$$M := \bigoplus_{m\in\mathbb{Z}} H^0(X, \mathcal{F}\otimes L_1^{\otimes m}), \quad N := \bigoplus_{n\in\mathbb{Z}} H^0(X, \mathcal{F}\otimes L_2^{\otimes n}),$$

$$P := \bigoplus_{m,n\in\mathbb{Z}} H^0(X, \mathcal{F}\otimes L_1^{\otimes m}\otimes L_2^{\otimes n}).$$

Thus C is a bigraded algebra and P is a bigraded C-module. Finite-dimensional truncations of P will be denoted by

$$P_{[(p_1,p_2),(q_1,q_2)]} = \bigoplus_{p_1 \le m \le q_1, p_2 \le n \le q_2} H^0(X, \mathcal{F} \otimes L_1^{\otimes m} \otimes L_2^{\otimes n}).$$

The choice of L_1, L_2 allows one to associate to any coherent sheaf $\mathcal{G}$ on X its Hilbert polynomial $\mathfrak{H}^{\mathcal{G}}(t,s)$ depending on two variables:

$$\mathfrak{H}^{\mathcal{G}}(t,s) = \dim H^0(X, \mathcal{G} \otimes L_1^{\otimes m} \otimes L_2^{\otimes n}) \quad \text{for } m,n \gg 0.$$

Let $\mathfrak{H}$ be a polynomial in $\mathbb{Q}[s,t]$ and let $Sub_{\mathfrak{H}}(\mathcal{F})$ be the part of the *Quot* scheme parametrizing subsheaves $\mathcal{K} \subset \mathcal{F}$ with $\mathfrak{H}^{\mathcal{K}} = \mathfrak{H}$. We have then the bigraded versions of the ordinary and of the C-Grassmannian:

$$G_C(\mathfrak{H}, P_{[(p_1,p_2),(q_1,q_2)]}) \subset G(\mathfrak{H}, P_{[(p_1,p_2),(q_1,q_2)]}),$$

and the following bigraded version of Theorem 1.4.1:

(4.4.3) Proposition. *For $0 \ll p_1 + p_2 \ll q_1 + q_2$, the image of the Grothendieck embedding $Sub_{\mathfrak{H}}(\mathcal{F}) \hookrightarrow G(\mathfrak{H}, P_{[(p_1,p_2),(q_1,q_2)]})$ is $G_C(\mathfrak{H}, P_{[(p_1,p_2),(q_1,q_2)]})$.*

The derived C-Grassmannian $RG_C(\mathfrak{H}, P_{[(p_1,p_2),(q_1,q_2)]})$ is defined as in §3. It is a dg-manifold (up to quasiisomorphism), and we again set

$$RSub_{\mathfrak{H}}(\mathcal{F}) = RG_C(\mathfrak{H}, P_{[(p_1,p_2),(q_1,q_2)]}).$$

Let now $\mathcal{K}$ be a subsheaf of $\mathcal{F}$ with $\mathfrak{H}^{\mathcal{K}} = \mathfrak{H}$. and let Z be the connected component of $Sub(\mathcal{F})$ containing $[\mathcal{K}]$. Set $h(s) = \mathfrak{H}(s,0)$ and $k(t) = \mathfrak{H}(0,t)$. Let Y_1 (respectively Y_2) be the component of $RSub_h(\mathcal{F})$ (respectively $RSub_k(\mathcal{F})$) containing Z in the decomposition given by Proposition (4.4.2).

(4.4.4) Proposition. *Y_1 and Y_2 are quasiisomorphic.*

Proof. Via the obvious embeddings $A \hookrightarrow C$ and $B \hookrightarrow C$, any C-module has also A- and B-module structures. This gives maps

$$RG_C(\mathfrak{H}, P_{[(p_1,0),(q_1,0)]}) \xrightarrow{\varrho_1} RG_A(h, M_{[p_1,q_1]}),$$

$$RG_C(\mathfrak{H}, P_{[(0,p_2),(0,q_2)]}) \xrightarrow{\varrho_2} RG_B(k, N_{[p_2,q_2]}).$$

By an obvious bigraded version of Theorem (4.3.2) (a), $RG_C(\mathfrak{H}, P_{[(p_1,0),(q_1,0)]})$ and $RG_C(\mathfrak{H}, P_{[(0,p_2),(0,q_2)]})$ are quasiisomorphic.

In general, the maps ϱ_1 and ϱ_2 induce at the level of π_0 maps which ar 1–1, but not surjections, since the decomposition of $Sub(\mathcal{F})$ indexed by polynomials in two variables is finer than the one indexed by polynomials in one variable. Using (4.4.2), we can replace $RG_C(\mathfrak{H}, P_{[(p_1,0),(q_1,0)]})$ and $RG_C(\mathfrak{H}, P_{[(0,p_2),(0,q_2)]})$ by quasi-isomorphic dg-schemes in which the connected components of π_0 are "separated". Let Z_1 and Z_2 be the (quasiisomorphic) respective components that contain $[\mathcal{K}]$. By shrinking Z_1 and Z_2 if necessary, we get induced maps $\varrho_i : Z_i \longrightarrow Y_i$, $i = 1,2$. The "Whitehead theorem" (2.5.9) implies now that ϱ_i are quasiisomorphisms. This completes the proof of Theorem 4.4.1.

References

[BG] A.K. Bousfield, V.K.A.M. Gugenheim, *On PL de Rham theory and rational homotopy type*, Memoirs AMS **179** (1976).

[CK] I. Ciocan-Fontanine, M. Kapranov, *Derived Hilbert schemes*, in preparation.

[Fu] W. Fulton, *Intersection Theory*, Springer-Verlag, 1984.

[Go] G. Gotzmann, *Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes*, Math. Zeit. **158** (1978), 61-70.

[Gr] A. Grothendieck, *Techniques de construction et théorèmes d'existence en géometrie algébrique IV: Les schémas de Hilbert*, Séminaire Bourbaki **221** (1960/61).

[GV] A. Grothendieck, J.-L. Verdier, *Préfaisceaux (SGA4, Exp. I)* Lecture Notes in Mathematics 269, (1972), Springer-Verlag, Berlin.

[Hi] V. Hinich, *Dg-coalgebras as formal stacks*, preprint math.AG/9812034.

[HMS] D. Husemoller, J. Moore, and J Stasheff, *Differential homological algebra and homogeneous spaces*, J. Pure Appl. Algebra **5** (1975), 113-185.

[Ka1] M. Kapranov, *Injective resolutions of BG and derived moduli spaces of local systems*, preprint alg-geom/9710027, to appear in J. Pure Appl. Alg.

[Ka2] M. Kapranov, *Rozansky-Witten invariants via Atiyah classes*, Compositio Math. **115** (1999), 71-113.

[Kol] J. Kollár, *Rational Curves on Algebraic Varieties*, Springer-Verlag, 1996.

[Kon] M. Kontsevich, *Enumeration of rational curves via torus actions*, in "*The Moduli Space of Curves*" (R. Dijkgraaf, C. Faber, G. van der Geer Eds.) Progress in Math. **129** (1995), Birkhauser, Boston, 335-368.

[Le] D. Lehmann, *Théorie homotopique des formes différentielles (d'après D. Sullivan)*, Asterique 45, (1977).

[Lo] J.-L. Loday, *Cyclic Homology*, Springer-Verlag, 1995.

[Ma] M. Markl, *A cohomology theory for A(m)-algebras and applications*, J. Pure Appl. Alg. **83** (1992), 141-175.

[Mc] J. McCleary, *User's Guide to Spectral Sequences,* Math. Lecture Ser., 12, Publish or Perish Inc., Wilmington DE, 1985.

[Mu] D. Mumford, *Lectures on curves on an algebraic surface*, Princeton Univ. Press, Princeton NJ, 1966.

[Q1] D. Quillen, *Homotopical Algebra,* Lecture Notes in Math. 43, Springer-Verlag, Berlin, 1967.

[Q2] D. Quillen, *Rational Homotopy theory*, Ann. Math. **90** (1969), 205-295.

[Re] C. Rezk, *Spaces of algebra structures and cohomology of operads*, thesis, MIT (1996).

[Se] J.-P. Serre, *Faisceaux algébriques cohérents*, Ann. of Math. **61** (1955), 197-278.

[Si] C. Simpson, *Descente pour les n-champs*, preprint math.AG/9807049.

[St1] J.D. Stasheff, *Homotopy associativity of H-spaces I, II*, Trans. Amer. Math. Soc. **108** (1963), 275-312.

[St2] J.D. Stasheff, *Differential graded Lie algebras, quasi-Hopf algebras and higher homotopy algebras*, Quantum Groups (P.P. Kulish Ed.) (Lecture Notes Math. **1510**), p. 120-137, Springer-Verlag, 1992.

[Vi] E. Viehweg, *Quasi-projective Moduli for Polarized Manifolds,* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 30, Springer-Verlag, Berlin, 1995.

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