FM 5012. Homework #1 (Due on Feb 28).

Consider the following variation of Newton's method for obtaining an approximation to $\sqrt{A}$:

(1.1a)   Take x^0 to be a positive number.

(1.1b)   Set $x^{k+1} = \alpha x^k + \beta A / x^k$ for $k \geq 0$.

(2pts) ① For what values of α and β is the method <u>consistent</u>?

(8pts) ② For what values of α and β is the method both <u>consistent and stable</u>?

(2pts) ③ For what values of α and β is the method convergent?

(2pts) ④ For what values of α and β does the method converge faster? Provide numerical results backing your answer.

(3pts) ⑤ Represent all elements of $\mathbb{R}(3,2)$.

(4pts) ⑥ Given any machine, how would you find the parameters s, t determining its set of numbers $\mathbb{R}(s,t)$? Provide numerical results supporting your idea.

Next, we want to consider our algorithm (1.11). For each ϵ and A, we are going to fill the following history of convergence table:

k	x^{k+1}	$x^{k+1} - x^k$	$(x^{k+1} - \sqrt{A}) / (x^{k+1} - x^k)$
0	...	...	...
1	...	...	...
2			
⋮			
N	...	...	...

(3pts) ⑦ Take $A = 5$ and $\epsilon = 10^{-8}$. Obtain the history of convergence table. Do the results match the theory? Now take $\epsilon = 10^{-6}$ and then 10^{-29}. Explain your results.

(3pts) ⑧ Repeat the above exercise with $A = 5 \cdot 10^{-6}$.

FM 5012   Homework #1:   A solution. (of first 4 questions)

We want to study the following variation of Newton's
method for obtaining an approximation to $\sqrt{A}$:

(1.1a)   Take x^0 to be a positive number.

(1.1b)   Set $x^{k+1} = \alpha x^k + \beta A / x^k$ for $k \geq 0$,

for some constants α and β. We want to know for
what values of these two constants the method (1.1)
is consistent, stable, and convergent. We also want to
know for what values it converges faster.

We say that the method is consistent if the exact
solution $\{x_e^k := \sqrt{A}\}_{k \geq 0}$ satisfies (1.1). It is not
difficult to see that (1.1a) is satisfied and that
(1.1b) is satisfied if and only if $\alpha + \beta = 1$.

Now let us consider the more difficult issue of the
stability of the method with respect to its initial
guess. By (1.1b), we have

$$X_1^{k+1} = \alpha X_1^k + \beta A / X_1^k \qquad \text{for } k \geq 0,$$
$$X_2^{k+1} = \alpha X_2^k + \beta A / X_2^k \qquad \text{for } k \geq 0,$$

and so, for $\delta^k := X_1^k - X_2^k$,

$$\delta^{k+1} = \alpha \delta^k + \beta A \left(\frac{1}{x_1^k} - \frac{1}{x_2^k} \right)$$

$$= \alpha \delta^k - \beta A \frac{\delta^k}{x_1^k x_2^k}$$

$$= \left(\alpha - \frac{\beta A}{x_1^k x_2^k} \right) \delta^k \qquad \text{for } k \geq 0.$$

To obtain a bound for the factor $(\alpha - \beta A / x_1^k x_2^k)$, we proceed as follows. First, note that α and β have to be nonnegative parameters. If not, if we take in (1.1a) the initial guess

$$x^0 = \sqrt{-\frac{\beta}{\alpha} A},$$

then $x^1 = 0$ and x^2 is not defined. (Note that we are assuming that the method is consistent, that is, that $\alpha + \beta = 1$. So, if α or β is negative, we immediately have that $\alpha \neq 0$ and $\beta / \alpha < 0$.)

This shows that we only need to consider the case $\beta = 1 - \alpha$ and $\alpha \in [0,1]$. If $\alpha = 0$, then $\beta = 1$ and we have that (1.1b) generates the sequence

$$x^0, \quad x^1 = \frac{A}{x^0}, \quad x^2 = x^0, \quad x^3 = \frac{A}{x^0}, \ldots .$$

This means that we have

$$\alpha - \frac{\beta A}{x_1^k x_2^k} = \begin{cases} -\dfrac{A}{x_1^0 x_2^0} & \text{for } k \text{ even,} \\[2mm] -\dfrac{x_1^0 x_2^0}{A} & \text{for } k \text{ odd,} \end{cases}$$

and so

$$\delta^1 = - \frac{A}{x_1^0 x_2^0} \, \delta^0$$

$$\delta^2 = - \frac{x_1^0 x_2^0}{A} \, \delta^1 = \delta^0$$

$$\delta^3 = - \frac{A}{x_1^0 x_2^0} \, \delta^2 = - \frac{A}{x_1^0 x_2^0} \, \delta^0$$

$$\delta^4 = \delta^0 ,$$

and so on. This implies that

$$|\delta^{k+1}| \le \max \left\{ 1, \frac{A}{x_1^0 x_2^0} \right\} |\delta^0| \qquad \text{for } k \ge 0 .$$

We thus see that the method is stable for $\alpha = 0$ and $\beta = 1$.

Next, we consider the case $\alpha \in (0, 1)$. In this case, we use the fact that, by (1.16),

$$x^{k+1} = \alpha \, x^k + \beta \frac{A}{x^k}$$

$$\ge \min_{x > 0} \left(\alpha x + \beta \frac{A}{x} \right)$$

$$= 2 \sqrt{\alpha \beta A} \qquad \text{for } k \ge 0 .$$

This immediately implies that

$$\left|\left(\alpha - \frac{\beta A}{X_1^k X_2^k}\right)\right| \in \left[\alpha - \frac{1}{4\alpha}, \alpha\right] \qquad \text{for } k \geqslant 1.$$

Hence

$$|\delta^{k+1}| \leq \max\left\{\alpha, \left|\alpha - \frac{1}{4\alpha}\right|\right\} |\delta^k| \qquad \text{for } k \geqslant 1,$$

$$|\delta^1| \leq \left|\alpha - \frac{\beta A}{X_1^0 X_2^0}\right| |\delta^0|$$

Set $\quad P = \max\left\{\alpha, \left|\alpha - \frac{1}{4\alpha}\right|\right\}$. Then we have that

$$|\delta^{k+1}| \leq P^k \left|\alpha - \frac{\beta A}{X_1^0 X_2^0}\right| |\delta^0| \qquad \text{for } k \geqslant 0.$$

To obtain stability, it is enough to require then that the parameter P be less than one. A little computation shows that this happens for $\alpha \in \left(\frac{\sqrt{2}-1}{2}, 1\right)$. Indeed, $P = 1$ if and only if $\alpha = 1$ or $\left|\alpha - \frac{1}{4\alpha}\right| = 1$. The latter case takes place if and only if

$$\frac{1}{4\alpha} - \alpha = 1,$$

that is, if and only if

$$4\alpha^2 + 4\alpha - 1 = 0$$

hence, if and only if $\alpha = \frac{\pm\sqrt{2}-1}{2}$. Since $\alpha > 0$, we take $\alpha = \frac{\sqrt{2}-1}{2}$. The parameter P is not bigger than one if α lies in $\left(\frac{\sqrt{2}-1}{2}, 1\right]$ as can be readily verified.

the method is thus stable for $\alpha \in \left(\frac{\sqrt{2}-1}{2}, 1 \right]$. Note that $\frac{1}{2}(\sqrt{2}-1) \approx 0.2$.

To obtain the stability of the method for $\alpha \in (0, \frac{\sqrt{2}-1}{2}]$ does not seem to be as simple as the previous cases.

Let us now deal with the issue of convergence. We thus use the fact that the method is consistent to replace $\{x_2^k\}_{k \geq 0}$ by the exact solution. Then dropping the subindex "2" from x_2^k, we obtain from our previous results on stability:

$$\left| \sqrt{A} - x^{k+1} \right| \leq \max \left\{ 1, \frac{\sqrt{A}}{x^0} \right\} \left| \sqrt{A} - x^0 \right| \qquad \text{for } k \geq 0$$

for $\alpha = 1$, and

$$\left| \sqrt{A} - x^{k+1} \right| \leq \rho^k \left| \alpha - \frac{(1-\alpha)\sqrt{A}}{x^0} \right| \left| \sqrt{A} - x^0 \right| \qquad \text{for } k \geq 0$$

and $\alpha \in (0, 1]$. From these results, we can conclude that the method converges for $\alpha \in \left(\frac{\sqrt{2}-1}{2}, 1 \right)$. Note that the method does not converge for $\alpha = 0$ and for $\alpha = 1$! With the approach we have used, we cannot say anything about the convergence of the method for $\alpha \in (0, \frac{\sqrt{2}-1}{2}]$.

Finally, let us consider for what values of α and β the method converges faster. Since we have

$$\sqrt{A} - x^{k+1} = \left(\alpha - \frac{\beta \sqrt{A}}{x^k} \right) \left(\sqrt{A} - x^k \right),$$

we can see that the error is reduced faster the smaller is the factor $(\alpha - \beta \sqrt{A} / x^k)$. If we assume that the sequence $\{x^k\}_{k \geq 0}$ converges to $\sqrt{A}$, such factor also converges to $(\alpha - \beta)$. This means that the smaller $|\alpha - \beta|$ the faster the convergence of the method should be. Clearly this happens when $\alpha = \beta$. Since $\beta = 1 - \alpha$, this implies that we expect the fastest convergence for $\alpha = \beta = \frac{1}{2}$.