Solution Guide for Homework #7

#4.B.3

Find all equilibrium solns of the system and determine stability

$$\begin{cases} \dot{x} = x^2 + y^2 - 1 \\ \dot{y} = 2xy \end{cases} \qquad (*)$$

Solution:

$\longrightarrow$ We firstly find equil. solns. Those are given by the ~~pairs~~ points (x_0, y_0), s.t:

$$\begin{cases} x_0^2 + y_0^2 - 1 = 0 \\ 2x_0 y_0 = 0 \end{cases} \Leftrightarrow \begin{cases} x_0^2 + y_0^2 = 1 \\ \left[\begin{array}{l} x_0 = 0 \\ y_0 = 0 \end{array}\right. \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x_0 = 0 \\ y_0 = 1 \end{cases} \\ \begin{cases} x_0 = 0 \\ y_0 = -1 \end{cases} \\ \begin{cases} x_0 = 1 \\ y_0 = 0 \end{cases} \\ \begin{cases} x_0 = -1 \\ y_0 = 0 \end{cases} \end{array}\right.$$

$\longrightarrow$ Assume (x_0, y_0) is an equil. point. Then, rewriting $(*)$ in terms of $v = x - x_0$ and $u = y - y_0$, we get:

$$\frac{d}{dt}\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} \frac{dv}{dx}\frac{dx}{dt} \\ \frac{du}{dy}\frac{dy}{dt} \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v^2 + 2vx_0 + x_0^2 + u^2 + 2uy_0 + y_0^2 - 1 \\ 2uv + 2vy_0 + 2ux_0 + 2x_0y_0 \end{pmatrix} =$$

$$= \begin{pmatrix} x_0^2 + y_0^2 - 1 \\ 2x_0y_0 \end{pmatrix} + \begin{pmatrix} 2x_0 & 2y_0 \\ 2y_0 & 2x_0 \end{pmatrix}\begin{pmatrix} v \\ u \end{pmatrix} + \text{terms of the higher order.}$$

$\to 0$

→ Thus we use Theorem 2 and Lemma 1 for $A = \begin{pmatrix} 2x_0 & 2y_0 \\ 2y_0 & 2x_0 \end{pmatrix}$ in order to determine the stability of each of the equil. points (x_0, y_0):

- $(0,1) \longrightarrow A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, so that $p(\lambda) = \lambda^2 - 4 = (\lambda-2)(\lambda+2)$ is char. polyn. of A and hence $\lambda_1 = -2, \lambda_2 = 2$ are eigenvalues. Since $Re(\lambda_2) = 2 > 0$, $(0,1)$ is <u>unstable</u> equil.

- $(0,-1) \longrightarrow A = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$, $p(\lambda) = \lambda^2 - 4 = (\lambda-2)(\lambda+2)$
$\lambda_1 = 2, \lambda_2 = 2$ - eigenvalues of A.
Since $Re(\lambda_2) = 2 > 0$, $(0,-1)$ is <u>unstable</u> equil.

- $(1,0) \longrightarrow A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $p(\lambda) = (\lambda-2)^2$
$\lambda_1 = \lambda_2 = 2$ - eigenvalues of A.
Since $Re(\lambda_1) = 2 > 0$, $(1,0)$ is <u>unstable</u> equil.

- $(-1,0) \longrightarrow A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$, $p(\lambda) = (\lambda+2)^2$
$\lambda_1 = \lambda_2 = -2$ - eigenvalues of A.
Since $Re(\lambda_1) = Re(\lambda_2) = -2 < 0$, $(-1,0)$ is <u>asymptotically stable</u> equil.

#4.4.7

Find the orbits of the system:

$$\begin{cases} \dot{x} = y(1+x+y) \\ \dot{y} = -x(1+x+y) \end{cases} \quad (*)$$

Solution:

$\longrightarrow$ We finstly determine equil. points of $(*)$:

$$\begin{cases} y_0(1+x_0+y_0)=0 \\ -x_0(1+x_0+y_0)=0 \end{cases} \Leftrightarrow \begin{bmatrix} 1+x_0+y_0=0 \\ \begin{cases} y_0=0 \\ -x_0(1+x_0+y_0)=0 \end{cases} \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1+x_0+y_0=0 \\ \begin{cases} x_0=0 \\ y_0=0 \end{cases} \end{bmatrix}$$

$\longrightarrow$ Now, for any non-equil. (x,y) we have $\left(\begin{smallmatrix}\text{since then}\\ 1+x+y\neq 0\end{smallmatrix}\right)$:

$$\frac{dx}{dy} = \frac{dx}{dt}\cdot\frac{dt}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{y(1+x+y)}{-x(1+x+y)} = -\frac{y}{x}$$

$$\Downarrow$$

$$-y\frac{dy}{dx} = x$$

$$\Updownarrow$$

$$\frac{d(-\frac{y^2}{2})}{dx} = \frac{d(\frac{x^2}{2})}{dx}$$

$$\Updownarrow$$

$$C + \frac{-y^2}{2} = \frac{x^2}{2}$$

$$\Updownarrow$$

$$\frac{x^2}{2} + \frac{y^2}{2} = C$$

Thus, (x_1, y_1) ← non-equil. lies on the orbit given by the connected component of the curve $\frac{x^2}{2} + \frac{y^2}{2} = C$ ($C > 0$, since $\frac{x_1^2}{2} + \frac{y_1^2}{2} \geq 0$ and if $\frac{x_1^2}{2} + \frac{y_1^2}{2} = 0$, (x_1, y_1) is an equil. point) with equilibrium points being removed.

→ Thus, orbits of (*) are:

- connected components of the curves $\frac{x_0^2}{2} + \frac{y_0^2}{2} = C$ $(C > 0)$ with points satisfying $x_0 + y_0 + 1 = 0$ or $(x_0, y_0) = 0$ removed.
- all the elements of the set $\{(x_0, y_0) \mid x_0 + y_0 + 1 = 0$ or $x_0 - y_0 = 0\}$.

In other words, orbits of (*) are:

- connected components of the circles $x^2 + y^2 = C$ $(C \geq 0)$ lying on the line with points (point) $x + y + 1 = 0$ removed
- each single point lying on the line $x + y + 1 = 0$

#4.6.11

Prove that all the solutions of the following equation are periodic: $\frac{d^2z}{dt^2} + \frac{z}{1+z^2} = 0$ (*)

Proof:

$\longrightarrow$ We notice that By Property 2 it suffices to show that orbits of (*) rewritten in the linear system form ($x = z$, $y = \frac{dz}{dt}$):

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{x}{1+x^2} \end{cases} \qquad (*)$$

are closed curves.

$\longrightarrow$ In order to find the orbits we firstly notice that $(x_0, y_0) = (0,0)$ is only equil. point of (*) and hence all other orbits of (*) are given by:

$$\frac{dy}{dx} = \frac{-\frac{x}{1+x^2}}{y} \Rightarrow \frac{d\left(\frac{y^2}{2}\right)}{dx} = \frac{d\left(-\frac{1}{2}\ln(1+x^2)\right)}{dx}$$

$$\Leftrightarrow \frac{y^2}{2} + \frac{1}{2}\ln(1+x^2) = C \Leftrightarrow$$

$$\Leftrightarrow y = \pm\sqrt{K - \ln(1+x^2)} \qquad (**)$$

Thus, every non-constant orbit consists of two parts defined as continious functions of x on the (connected) domain $K - \ln(1+x^2) \geq 0$ and given by:

$y = \sqrt{K - en(1+x^2)}$, $y = -\sqrt{K - en(1+x^2)}$. Both of them intersect x-axis at the same points. Hence, as by Property 1 orbits can't be self-intersecting, the union of these curves (that is every non-constant orbits) ~~are~~ is a simple closed curve.

Same obviously holds for any constant orbit.
This completes the proof. ▨