Homework #1

1. $$\frac{dy}{dt} + \sqrt{1+t^2}\, y = 0 \qquad t>0$$
$$y(0) = \sqrt{5}$$

the equation is separable and so we can write that

$$\frac{1}{y}\frac{dy}{dt} = -\sqrt{1+t^2}$$

Since $\frac{1}{y}\frac{dy}{dt} = \frac{d}{dt}\ln|y|$, we get that

$$\int_0^t \frac{d}{ds}\ln|y(s)|\,ds = -\int_0^t \sqrt{1+s^2}\,ds$$

$$\Rightarrow \quad \ln|y(t)| - \ln|\sqrt{5}| = -\int_0^t \sqrt{1+s^2}\,ds$$

$$\Rightarrow \quad |y(t)| = \sqrt{5}\exp\left(-\int_0^t \sqrt{1+s^2}\,ds\right)$$

$$\Rightarrow \quad y(t) = \sqrt{5}\exp\left(-\int_0^t \sqrt{1+s^2}\,ds\right).$$

It remains to calculate the expression $\theta(t) := -\int_0^t \sqrt{1+s^2}\,ds$.
One way to do that is to write

$$s = \frac{e^{x} - e^{-x}}{2}.$$

then, when $s=0$, we have that $x=0$. when $s=t$, we have that

$$t = \frac{e^{x(t)} - e^{-x(t)}}{2}$$

$$(*) \quad \Rightarrow e^{2x(t)} - 2t\,e^{x(t)} - 1 = 0$$

$$\Rightarrow e^{x(t)} = t + \sqrt{t^2+1} \quad \Rightarrow x(t) = \ln\left(t+\sqrt{t^2+1}\right).$$

Moreover,

$$ds = \frac{e^{x} + e^{-x}}{2}\, dx$$

$$\sqrt{1+s^2} = \sqrt{1 + \left(\frac{e^{x} - e^{-x}}{2}\right)^2} = \frac{e^{x}+e^{-x}}{2}.$$

then

$$\theta(t) = -\int_0^t \sqrt{1+s^2}\, ds$$

$$= -\int_0^{x(t)} \left(\frac{e^{x}+e^{-x}}{2}\right)^2 dx$$

$$= -\int_0^{x(t)} \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) dx$$

$$= -\frac{1}{4}\left(\frac{e^{2x}}{2}\Big|_0^{x(t)} + 2x(t) - \frac{e^{-2x}}{2}\Big|_0^{x(t)}\right)$$

$$= -\frac{1}{4}\left(\frac{1}{2}e^{2x(t)} - \cancel{\frac{1}{2}} + 2x(t) - \frac{1}{2}e^{-2x(t)} + \cancel{\frac{1}{2}}\right)$$

$$= -\frac{1}{8}e^{2x(t)} - \frac{1}{2}x(t) + \frac{1}{8}e^{-2x(t)}$$

$$= -\frac{1}{8}\left(2t\, e^{x(t)} + \cancel{1}\right) - \frac{1}{2}x(t) + \frac{1}{8}\left(\cancel{1} - 2t\, e^{-x(t)}\right)$$

by using equation (*). then

$$\theta(t) = -\frac{t}{4}\left(e^{x(t)} + e^{-x(t)}\right) - \frac{1}{2}x(t)$$

$$= -\frac{t}{4}\left(t + \sqrt{t^2+1} + \frac{1}{t+\sqrt{t^2+1}}\right) - \frac{1}{2}\ln\left(t+\sqrt{t^2+1}\right)$$

by using the definition of $x(t)$. then

$$\theta(t) = -\frac{t}{4}\,\frac{2t^2+2+2t\sqrt{t^2+1}}{t+\sqrt{t^2+1}} - \frac{1}{2}\ln\left(t+\sqrt{t^2+1}\right)$$

$$= -\frac{t\sqrt{t^2+1}}{2} - \frac{1}{2}\ln\left(t+\sqrt{t^2+1}\right)$$

This implies that

$$y(t) = \sqrt{5}\,\exp\left(-\frac{t\sqrt{t^2+1}}{2} - \frac{1}{2}\ln\left(t+\sqrt{t^2+1}\right)\right)$$

$$= \sqrt{5}\,\exp\left(-\frac{t\sqrt{t^2+1}}{2}\right)\frac{1}{\sqrt{t+\sqrt{t^2+1}}}.$$

2 $\begin{cases} \dfrac{dy}{dt} + ty = 1+t \\ y(3/2) = 0 \end{cases}$

We know that the integrating factor is $\mu(t) = e^{t^2/2}$ since $\mu'(t) = t\,e^{t^2/2} = t\mu(t)$. Then we have that

$$\frac{d}{dt}\left(\mu(t)\,y(t)\right) = (1+t)\,\mu(t)$$

and so, integrating on t from 3/2 to "t", we get

$$e^{t^2/2}\,y(t) = \int_{3/2}^{t}(1+s)\,e^{s^2/2}\,ds$$

$$\Rightarrow\ y(t) = e^{-t^2/2}\int_{3/2}^{t}(1+s)\,e^{s^2/2}\,ds$$

$$= e^{-t^2/2}\left(\int_{3/2}^{t} e^{s^2/2}\,ds + e^{s^2/2}\Big|_{3/2}^{t}\right)$$

$$= e^{-t^2/2}\int_{3/2}^{t} e^{s^2/2}\,ds + 1 - e^{-t^2/2+9/8}.$$

3 $\begin{cases} \dfrac{dy}{dt} = \dfrac{2t}{y+yt^2} \\ y(2) = 3 \end{cases}$

Again the equation is separable and we can write that

$$y\frac{dy}{dt} = \frac{2t}{1+t^2}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}y^2\right) = \frac{d}{dt}\ln(1+t^2)$$

Integrating, we get that

$$\frac{1}{2}y^2(t) - \frac{1}{2}9 = \ln\frac{1+t^2}{5}$$

$$\Rightarrow \quad y(t) = \sqrt{9 + 2\ln\left(\frac{1+t^2}{5}\right)}.$$

4 $\dfrac{dy}{dt} = K(a-y)(b-y)$

$y(0) = 0$

where $a, b > 0$.

Let us assume that $a < b$. Then

$$\frac{1}{(a-y)(b-y)}\frac{dy}{dt} = K$$

$$\Rightarrow \quad \frac{A}{a-y} + \frac{B}{b-y} = \frac{1}{(a-y)(b-y)} \quad \Leftrightarrow \quad \begin{cases} Ab + aB = 1 \\ A + B = 0 \end{cases}$$

$$\Leftrightarrow \quad A = \frac{1}{b-a} = -B.$$

$$\Rightarrow \left(\frac{1}{a-y} - \frac{1}{b-y}\right)\frac{dy}{dt} = k(b-a)$$

$$\Rightarrow \frac{d}{dt} \ln\left|\frac{b-y}{a-y}\right| = k(b-a)$$

Integrating in "t", we get

$$\ln\left|\frac{b-y(t)}{a-y(t)}\right| - \ln\left|\frac{b}{a}\right| = tk(b-a)$$

$$\Rightarrow \left|\frac{b-y(b)}{a-y(t)}\right| = \left|\frac{b}{a}\right| \exp(k(b-a)t)$$

$$\Rightarrow \frac{b-y(t)}{a-y(t)} = \frac{b}{a}\exp(k(b-a)t)$$

$$\Rightarrow y(t) = \frac{b - b\exp(k(b-a)t)}{1 - \frac{b}{a}\exp(k(b-a)t)} .$$

Let us now assume that $a=b$. then

$$\frac{1}{(a-y)^2}\frac{d}{dt}y = k$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{a-y}\right) = k$$

$$\Rightarrow \frac{1}{a-y(t)} - \frac{1}{a} = kt$$

$$\Rightarrow \frac{1}{a-y(t)} = \frac{akt+1}{a}$$

$$\Rightarrow y(t) = a - \frac{a}{akt+1}$$

$$= \frac{a^2kt}{akt+1} .$$