Math 4512 Homework #3

2.2 Q5.

$$y'' - 3y' - 4y = 0; \quad y(0) = 1, \; y'(0) = 0$$

Solution.

The characteristic equation is

$$r^2 - 3r - 4 = 0$$

The two roots are

$$r_1 = -1 \quad \text{and} \quad r_2 = 4$$

The fundamental set of solutions is

$$y_1(t) = e^{-t} \quad y_2(t) = e^{4t}$$

$$\Rightarrow \quad y(t) = c_1 e^{-t} + c_2 e^{4t}$$

By $y(0) = 1$, $c_1 + c_2 = 1$ ①

By $y'(0) = 0$, $-c_1 + 4c_2 = 0$ ②

① and ② $\Rightarrow$ $c_1 = \frac{4}{5}$, $c_2 = \frac{1}{5}$

Therefore,

$$y(t) = \frac{4}{5} e^{-t} + \frac{1}{5} e^{4t}$$

$\square$

2.2.1 Q5.

$$y'' + y' + 2y = 0 \ ; \quad y(0) = 1, \ y'(0) = -2$$

Solution.

The characteristic equation is

$$r^2 + r + 2 = 0$$

The two roots are

$$r_1 = \frac{-1+\sqrt{1-8}}{2} = -\frac{1}{2} + \frac{\sqrt{7}}{2}i$$

and

$$r_2 = \frac{-1-\sqrt{1-8}}{2} = -\frac{1}{2} - \frac{\sqrt{7}}{2}i$$

$$e^{r_1 t} = e^{(-\frac{1}{2}+\frac{\sqrt{7}}{2}i)t} = e^{-\frac{t}{2}}\left(\cos\frac{\sqrt{7}}{2}t + i\sin\frac{\sqrt{7}}{2}t\right)$$

$$\Rightarrow \operatorname{Re}\{e^{r_1 t}\} = e^{-\frac{t}{2}}\cos\frac{\sqrt{7}}{2}t \ \text{ and } \ \operatorname{Im}\{e^{r_1 t}\} = e^{-\frac{t}{2}}\sin\frac{\sqrt{7}}{2}t$$

Consequently,

$$y(t) = e^{-\frac{t}{2}}\left[C_1\cos\frac{\sqrt{7}}{2}t + C_2\sin\frac{\sqrt{7}}{2}t\right]$$

$$y(0) = 1 \Rightarrow C_1 = 1 \qquad ①$$

$$y'(0) = -2 \Rightarrow -\frac{1}{2}C_1 + \frac{\sqrt{7}}{2}C_2 = -2 \qquad ②$$

① and ② $\Rightarrow$ $C_1 = 1$ and $C_2 = -\frac{3\sqrt{7}}{7}$

Therefore,

$$y(t) = e^{-\frac{t}{2}}\left[\cos\frac{\sqrt{7}}{2}t - \frac{3\sqrt{7}}{7}\sin\frac{\sqrt{7}}{2}t\right]$$

2.2.2 Q5.

Method 1 :

$$y_1(t) = e^{-b(t-t_0)/2a}$$

$$= ke^{-bt/2a} \quad \text{where } k = e^{bt_0/2a}$$

$$y_1'(t) = \frac{-b}{2a} k e^{-bt/2a}$$

$$y_1''(t) = \frac{b^2}{4a^2} k e^{-bt/2a}$$

$$a y_1''(t) + b y_1'(t) + c y_1(t) \quad (1)$$

$$= \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right) k e^{-bt/2a}$$

$$= (c - 2c + c) k e^{-bt/2a} \quad (\text{by } b^2 = 4ac)$$

$$= 0$$

Thus, $y_1(t)$ is a solution of (1).

$$y_2(t) = (t - t_0) e^{-b(t-t_0)/2a}$$

$$= kte^{-bt/2a} - t_0 k e^{-bt/2a}$$

$$y_2'(t) = \left(1 - \frac{b}{2a}t + \frac{bt_0}{2a}\right) k e^{-bt/2a}$$

$$y_2''(t) = \left(-\frac{b}{a} + \frac{b^2}{4a^2}t - \frac{b^2 t_0}{4a^2}\right) k e^{-bt/2a}$$

$$a y_2''(t) + b y_2'(t) + c y_2(t) \quad (1)$$

$$= \left(-\frac{b^2}{4a}t + \frac{t_0 b^2}{4a} + ct - ct_0\right) k e^{-bt/2a}$$

$$= (-ct + ct_0 + ct - ct_0) k e^{-bt/2a} = 0 \quad (\text{by } b^2 = 4ac)$$

Thus, $y_2(t)$ is a solution of (1)

Method 2:

$$ay'' + by' + cy = 0 \quad \text{where } b^2 = 4ac$$

$\Rightarrow$ the root r is $-\frac{b}{2a}$

One solution is $\tilde{y}(t) = e^{-\frac{bt}{2a}}$

$y_1(t) = ke^{-bt/2a}$ where $k = e^{bt_0/2a}$ is a constant.

So $y_1(t)$ is a solution. By the method from §2.2.2, we know that a second solution $\bar{y}(t)$ is given by

$$\bar{y}(t) = \tilde{y}(t)\, v(t) = \tilde{y}(t) \int u(t)\, dt$$

where $u(t) = \frac{dv(t)}{dt} = \frac{\exp(-\int \frac{b}{a} dt)}{\tilde{y}^2(t)} = 1$

Hence, $\bar{y}(t) = \tilde{y}(t) \int dt = te^{-bt/2a}$

$y_2(t) = kte^{\frac{-bt}{2a}} - kt_0 e^{\frac{-bt}{2a}}$

Notice that $kte^{\frac{-bt}{2a}}$ is a solution, $kt_0 e^{\frac{-bt}{2a}}$ is also a solution. So the linear combination of the two, which is $y_2(t)$, is a solution.

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2.3 Q3

The functions

$$\psi_2(t) - \psi_1(t) = 4e^t$$

and

$$\psi_3(t) - \psi_2(t) = -2e^t + e^{-t^3}$$

are solutions of the corresponding homogeneous equation. Moreover, these functions are linearly independent. Therefore, every solution $y(t)$ of this equation must be of the form

$$y(t) = 4c_1 e^t + c_2 e^{-t^3} - 2c_2 e^t + \underbrace{3e^t + e^{t^2}}_{\psi_1(t)}$$

$y(0) = 1 \Rightarrow 4c_1 - c_2 + 4 = 1$ ①

$y'(0) = 2 \Rightarrow 4c_1 - 2c_2 + 3 = 2$ ②

① and ② $\Rightarrow c_1 = -\frac{5}{4}$ and $c_2 = -2$

Therefore, the solution is given by

$$y(t) = 2e^t - 2e^{-t^3} + e^{t^2}$$