Math 4512 HW6 Solution

§3.8 12

$$\dot{x} = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{pmatrix} x$$

$$P(\lambda) = \det\begin{pmatrix} 3-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 4 & 1 & -3-\lambda \end{pmatrix} = (3-\lambda)[(\lambda-2)(\lambda+3)-1] - [(3+\lambda)-4] - 2[-1-4(2-\lambda)]$$

$$= (\lambda-2)(\lambda+1)(1-\lambda)$$

Eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = -1$

(i) $\lambda_1 = 2$

$$\begin{pmatrix} 1 & 1 & -2 \\ -1 & 0 & 1 \\ 4 & 1 & -5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow x^1(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(ii) $\lambda_2 = 1$

$$\begin{pmatrix} 2 & 1 & -2 \\ -1 & 1 & 1 \\ 4 & 1 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = c_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow x^2(t) = c_2 e^{t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

(iii)

$$\lambda_3 = -1$$

$$\begin{pmatrix} 4 & 1 & -2 \\ -1 & 3 & 1 \\ 4 & 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = C_3 \begin{pmatrix} 7 \\ -2 \\ 13 \end{pmatrix}$$

$$\Rightarrow X^3(t) = C_3 e^{-t} \begin{pmatrix} 7 \\ -2 \\ 13 \end{pmatrix}$$

(i)(ii)(iii) $\Rightarrow X(t) = C_1 e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_3 e^{-t}\begin{pmatrix} 7 \\ -2 \\ 13 \end{pmatrix}$

$$X(0) = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} C_1 + C_2 + 7C_3 \\ C_1 - 2C_3 \\ C_1 + C_2 + 13C_3 \end{pmatrix} \Rightarrow C_1 = 6,\ C_2 = -12,\ C_3 = 1$$

$$\Rightarrow X(t) = 6e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 12e^{t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} 7 \\ -2 \\ 13 \end{pmatrix}$$

§ 3.9 7

$$\dot{X} = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} X$$

$$P(\lambda)=\begin{pmatrix}-3-\lambda & 0 & 2\\ 1 & -1-\lambda & 0\\ -2 & -1 & -\lambda\end{pmatrix}=(-3-\lambda)\lambda(1+\lambda)+2(-1-2(1+\lambda))$$

$$=(\lambda+2)(\lambda^2+2\lambda+3)$$

$$\Rightarrow \lambda_1=-2,\ \lambda_2=\frac{-2+\sqrt{-8}}{2}=-1+\sqrt{2}i,\ \lambda_3=-1-\sqrt{2}i$$

(i) $\lambda_1=-2$

$$\begin{pmatrix}-1 & 0 & 2\\ 1 & 1 & 0\\ -2 & -1 & 2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}2\\-2\\1\end{pmatrix}c_1$$

$$\Rightarrow X^1(t)=c_1e^{-2t}\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

(ii) $\lambda_2=-1+\sqrt{2}i$

$$\begin{pmatrix}-2-\sqrt{2}i & 0 & 2\\ 1 & -\sqrt{2}i & 0\\ -2 & -1 & 1-\sqrt{2}i\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}2\\-\sqrt{2}i\\2+\sqrt{2}i\end{pmatrix}$$

$$e^{(-1+\sqrt{2}i)t}\begin{pmatrix}2\\-\sqrt{2}i\\2+\sqrt{2}i\end{pmatrix}=e^{-t}(\cos\sqrt{2}t+i\sin\sqrt{2}t)\left[\begin{pmatrix}2\\0\\2\end{pmatrix}+i\begin{pmatrix}0\\-\sqrt{2}\\\sqrt{2}\end{pmatrix}\right]$$

$$=e^{-t}\left[\cos\sqrt{2}t\begin{pmatrix}2\\0\\2\end{pmatrix}+i\cos\sqrt{2}t\begin{pmatrix}0\\-\sqrt{2}\\\sqrt{2}\end{pmatrix}+i\sin\sqrt{2}t\begin{pmatrix}2\\0\\2\end{pmatrix}\right.$$

$$\left.-\sin\sqrt{2}t\begin{pmatrix}0\\-\sqrt{2}\\\sqrt{2}\end{pmatrix}\right]$$

$$\Rightarrow \quad x^2(t) = e^{-t}\begin{pmatrix} 2\cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \\ 2\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{pmatrix}$$

$$x^3(t) = e^{-t}\begin{pmatrix} 2\sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \\ \sqrt{2}\cos\sqrt{2}t + 2\sin\sqrt{2}t \end{pmatrix}$$

$$\Rightarrow \quad X(t) = C_1 X^1(t) + C_2 X^2(t) + C_3 X^3(t)$$

$$X(0) = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2C_1 + 2C_2 \\ -2C_1 - \sqrt{2}C_3 \\ C_1 + 2C_2 + \sqrt{2}C_3 \end{pmatrix} \Rightarrow C_1 = 1,\ C_2 = -1,\ C_3 = -\frac{\sqrt{2}}{2}$$

$$\Rightarrow \quad X(t) = e^{-2t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} -2\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t + \cos\sqrt{2}t \\ -3\cos\sqrt{2}t \end{pmatrix}$$

§3.10 12

(a) By Q11.

$$e^{At} = e^{2t}\left[I + t\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ 0 & & & 0 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & & 0 & 0 \\ 0 & & & 0 \end{pmatrix} + \frac{t^3}{3!}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right]$$

$$= e^{2t}\begin{pmatrix} 1 & t & t^2/2! & t^3/3! \\ & 1 & t & t^2/2! \\ & & 1 & t \\ 0 & & & 1 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 2I + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e^{At} = e^{2t} \exp\left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} t \right]$$

$$= e^{2t} \left[I + t \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{t^2}{2!} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right]$$

$$= e^{2t} \begin{pmatrix} 1 & t & t^2/2! & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(c)

By (b),

$$e^{At} = e^{2t} \left[I + t \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right]$$

$$= e^{2t} \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

§ 3.12 5

$$\dot{x} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}, \quad x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \quad P(\lambda) = \begin{pmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{pmatrix} = (2-\lambda)^2(3-\lambda)$$

$$\Rightarrow \lambda_1 = \lambda_2 = 2, \; \lambda_3 = 3$$

(i) $\lambda_1 = \lambda_2 = 2$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x^1(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^2\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$x^2(t) = e^{2t}[I + t(A-2I)]\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$= e^{2t}\left[I + t\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\right]\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = e^{2t}\left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\right]$$

$$= e^{2t}\begin{pmatrix} -t \\ 1 \\ -1 \end{pmatrix}$$

(iii) $\lambda_3 = 3$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$X^3(t) = e^{3t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

We guess a particular solution $\psi(t) = be^{2t}$

Plugging in, we have

$$2be^{2t} = Abe^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}$$

$$\Rightarrow (A-2I)b = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} b = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow X(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} -t \\ 1 \\ -1 \end{pmatrix} + c_3 e^{3t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + e^{2t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$X(0) = \begin{pmatrix} c_1 + c_3 + 1 \\ c_2 \\ -c_2 + c_3 - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{array}{l} c_1 = -3 \\ c_2 = 1 \\ c_3 = 3 \end{array}$$

$$\Rightarrow X(t) = e^{2t}\begin{pmatrix} -2-t \\ 1 \\ -2 \end{pmatrix} + 3e^{3t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$