§4.3 2

$$\dot{x} = x^2 + y^2 - 1$$
$$\dot{y} = x^2 - y^2$$

$\begin{cases} x^2+y^2-1=0 \\ x^2-y^2=0 \end{cases} \Rightarrow$ Equibrium solutions are $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$
$(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$

① $x(t)=\frac{\sqrt{2}}{2}$, $y(t)=\frac{\sqrt{2}}{2}$. Set $u=x-\frac{\sqrt{2}}{2}$, $v=y-\frac{\sqrt{2}}{2}$ Then

$$\frac{du}{dt} = \frac{dx}{dt} = (u+\frac{\sqrt{2}}{2})^2 + (v+\frac{\sqrt{2}}{2})^2 - 1$$
$$= \sqrt{2}u + \sqrt{2}v + (u^2+v^2)$$

$$\frac{dv}{dt} = \frac{dy}{dt} = (u+\frac{\sqrt{2}}{2})^2 - (v+\frac{\sqrt{2}}{2})^2$$
$$= \sqrt{2}u - \sqrt{2}v + (u^2-v^2)$$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u^2+v^2 \\ u^2-v^2 \end{pmatrix}$$

The matrix $\begin{pmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}$ has $\lambda_1 = 2$ $\lambda_2 = -2$

$\Rightarrow$ $x(t)=\frac{\sqrt{2}}{2}$ $y(t)=\frac{\sqrt{2}}{2}$ is $\boxed{\text{unstable}}$.

② $x(t)=\frac{\sqrt{2}}{2}$ $y(t)=-\frac{\sqrt{2}}{2}$ Set $u=x-\frac{\sqrt{2}}{2}$ $v=y+\frac{\sqrt{2}}{2}$

Then $\frac{du}{dt}=(u+\frac{\sqrt{2}}{2})^2+(v-\frac{\sqrt{2}}{2})^2-1=\sqrt{2}u-\sqrt{2}v+(u^2+v^2)$

$$\frac{dv}{dt}=(u+\frac{\sqrt{2}}{2})^2-(v-\frac{\sqrt{2}}{2})^2=\sqrt{2}u+\sqrt{2}v+(u^2-v^2)$$

$$\frac{d}{dt}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}\sqrt{2}&-\sqrt{2}\\\sqrt{2}&\sqrt{2}\end{pmatrix}+\begin{pmatrix}u^2+v^2\\u^2-v^2\end{pmatrix}$$

The matrix $\begin{pmatrix}\sqrt{2}&-\sqrt{2}\\\sqrt{2}&\sqrt{2}\end{pmatrix}$ has $\lambda_1=\sqrt{2}-\sqrt{2}i$ $\lambda_2=\sqrt{2}+\sqrt{2}i$.

Hence, $x(t)=\frac{\sqrt{2}}{2}$, $y(t)=-\frac{\sqrt{2}}{2}$ is $\boxed{\text{unstable}}$.

③ $x(t)=-\frac{\sqrt{2}}{2}$, $y(t)=\frac{\sqrt{2}}{2}$ Set $u=x+\frac{\sqrt{2}}{2}$ $v=y-\frac{\sqrt{2}}{2}$

Then $\frac{du}{dt}=(u-\frac{\sqrt{2}}{2})^2+(v+\frac{\sqrt{2}}{2})^2-1=-\sqrt{2}u+\sqrt{2}v+(u^2+v^2)$

$$\frac{dv}{dt}=(u-\frac{\sqrt{2}}{2})^2-(v+\frac{\sqrt{2}}{2})^2=-\sqrt{2}u-\sqrt{2}v+(u^2-v^2)$$

$$\frac{d}{dt}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}-\sqrt{2}&\sqrt{2}\\-\sqrt{2}&-\sqrt{2}\end{pmatrix}+\begin{pmatrix}u^2+v^2\\u^2-v^2\end{pmatrix}$$

The matrix $\begin{pmatrix}-\sqrt{2}&\sqrt{2}\\-\sqrt{2}&-\sqrt{2}\end{pmatrix}$ has $\lambda_1=-\sqrt{2}+\sqrt{2}i$ $\lambda_2=-\sqrt{2}-\sqrt{2}i$

Hence, $x(t)=-\frac{\sqrt{2}}{2}$, $y(t)=\frac{\sqrt{2}}{2}$ is $\boxed{\text{stable}}$.

④ $x(t) = -\frac{\sqrt{2}}{2}$, $y(t) = -\frac{\sqrt{2}}{2}$ Set $u = x + \frac{\sqrt{2}}{2}$ $v = y + \frac{\sqrt{2}}{2}$

Then $\frac{du}{dt} = (u - \frac{\sqrt{2}}{2})^2 + (v - \frac{\sqrt{2}}{2})^2 - 1 = -\sqrt{2}u - \sqrt{2}v + (u^2 + v^2)$

$$\frac{dv}{dt} = (u - \frac{\sqrt{2}}{2})^2 - (v - \frac{\sqrt{2}}{2})^2 = -\sqrt{2}u + \sqrt{2}v + (u^2 - v^2)$$

$$\frac{d}{dt}\begin{pmatrix} v \\ v \end{pmatrix} = \begin{pmatrix} -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{pmatrix} + \begin{pmatrix} u^2 + v^2 \\ u^2 - v^2 \end{pmatrix}$$

The matrix $\begin{pmatrix} -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{pmatrix}$ has $\lambda_1 = 2$ $\lambda_2 = -2$

Hence, $x(t) = -\frac{\sqrt{2}}{2}$ $y(t) = -\frac{\sqrt{2}}{2}$ is $\boxed{\text{unstable}}$

$\blacksquare$

§4.4 7

$$\dot{x} = y(1 + x + y)$$

$$\dot{y} = -x(1 + x + y)$$

When $1 + x + y \neq 0$ and $x \neq y \neq 0$.

$\frac{dy}{dx} = -\frac{x}{y}$ $\Rightarrow$ The orbit is $x^2 + y^2 = c$ $(c > 0)$

$x = y = 0$ and $1 + x + y$ are also orbits

Hence, the orbits are $x^2 + y^2 = c$ $(c \geq 0)$ minus $x + y + 1 = 0$

and $x + y + 1 = 0$.

$$m\ddot{z} + c\dot{z} + k\dot{z} = 0$$

$$\ddot{z} + \frac{c}{m}\dot{z} + \frac{k}{m}\dot{z} = 0 \Rightarrow \lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km}$$

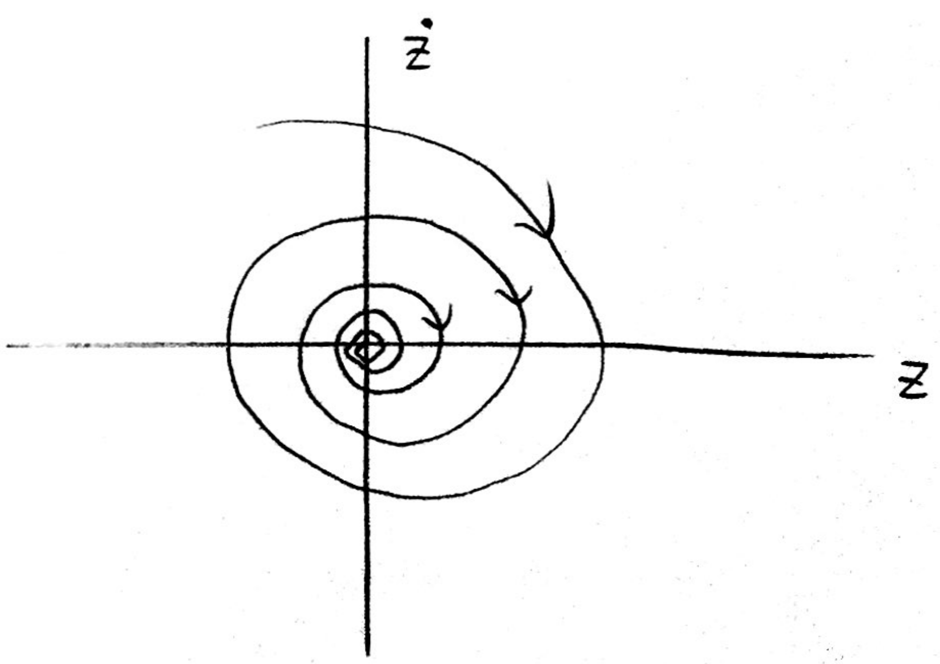
$$x_1 = z,\ x_2 = \dot{z} \Rightarrow x' = Ax,\ A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}$$

Underdamped Case : $0 < c^2 < 4km$

$\lambda_{1,2} = -\alpha \pm \omega i$ where $\alpha = c/2m$

$\omega = \sqrt{4km - c^2}/2m$

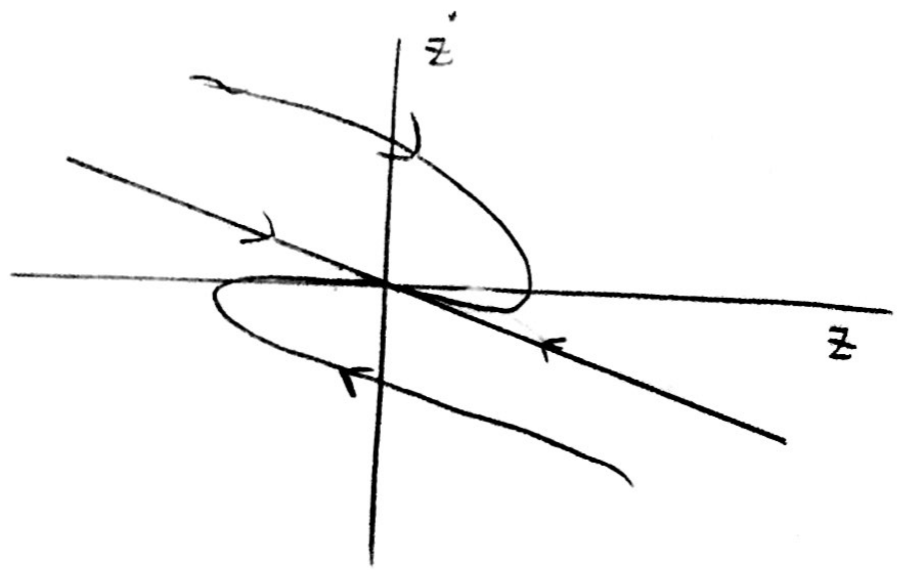
$$y(t) = e^{-\alpha t}(C_1 \cos(\omega t) + C_2 \sin(\omega t))$$



Critically Damped Case : $c^2 = 4km$

$\lambda_1 = \lambda_2 = -\mu/2m$

$y(t) = e^{\lambda_1 t}(c_1 + c_2 t)$



Overdamped Case : $c^2 > 4km$

$\lambda_1 < \lambda_2 < 0$

$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$