First MATH 4512 mid-term exam: Friday Sept. 30, 2016 NAME:

(Each problem is 5 points)

1. Find the solution of the initial-value problem $\frac{d}{dt}y = 2ty + t$ and $y(0) = 1$.

This is a separable equation since we can write $\frac{d}{dt}y = (2y+1)t$. Assuming that $2y+1 \neq 0$, we get that $\frac{1}{2y+1}\frac{dy}{dt} = t$, which implies that

$\frac{d}{dt}\left(\frac{1}{2}\ln|2y+1|\right) = t$. A simple integration gives us that

$$\frac{1}{2}\ln|2y(t)+1| - \frac{1}{2}\ln|2\cdot 1+1| = \frac{1}{2}t^2$$

this implies that

$$\ln\left|\frac{1}{3}(2y(t)+1)\right| = t^2$$

$$\Rightarrow \left|\frac{1}{3}(2y(t)+1)\right| = e^{t^2}$$

(Note that this implies that $2y+1$ is never equal to zero, as assumed above.) the choice consistent with the initial condition is

$$\frac{1}{3}(2y(t)+1) = e^{t^2}$$

which implies that

$$y(t) = (3e^{t^2}-1)/2.$$

2. Find the solution of the initial-value problem $\frac{d}{dt}y = 1 - y^2$ and $y(0) = 0$. What is the limit of the solution when t goes to infinity?

Assuming that $y^2 \neq 1$, we get

$$\frac{1}{1-y^2}\frac{dy}{dt} = 1.$$

Since

$$\frac{1}{1-y^2} = \frac{1}{2}\frac{1}{1+y} + \frac{1}{2}\frac{1}{1-y},$$

we get that

$$\frac{1}{2}\frac{1}{1+y}\frac{dy}{dt} + \frac{1}{2}\frac{1}{1-y}\frac{dy}{dt} = 1$$

or, that

$$\frac{d}{dt}\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| = 1.$$

Integrating, we get

$$\ln\left|\frac{1+y(t)}{1-y(t)}\right| - \ln\left|\frac{1+0}{1-0}\right| = 2t$$

$$\Rightarrow \left|\frac{1+y(t)}{1-y(t)}\right| = e^{2t}.$$

(Note that this is consistent with our assumption that $y^2 \neq 1$.) the only choice satisfying the initial condition is

$$\frac{1+y(t)}{1-y(t)} = e^{2t},$$

which implies that

$$y(t) = \frac{e^{2t}-1}{e^{2t}+1}.$$

As t goes to infinity, we see that $y(t)$ goes to 1. this could have been obtained by taking a simple look to the phase diagram:

($y=1$ and $y=-1$ are equilibrium points).

3. State the theorem of the existence and uniqueness of the solution of the initial-value problem $\frac{d}{dt}y = f(y,t)$ and $y(t_0) = y_0$. When $f(y,t) = |y|^{1/2}, t_0 = 0$ and $y_0 = 0$ show that there are two solutions and argue why this does not contradict the theorem.

Assume that $f(y,t)$ and $\partial_y f(y,t)$ are continuous on the rectangle

$$R = [y_0 - b, y_0 + b] \times [t_0, t_0 + a]$$

Then, our initial-value problem has a unique solution for t in

$$[t_0, t_0 + \alpha]$$

where $\alpha = \min\{a, \frac{b}{M}\}$, where $M = \max_{(y,t) \in R} |f(y,t)|$.

When $f(y,t) = |y|^{1/2}$, $t_0 = 0$ and $y_0 = 0$, it is easy to see that $y(t) = 0$ and $y(t) = \frac{t^2}{4}$ are solutions. However, the theorem is not contradicted because $\frac{\partial}{\partial y} f(y,t) = \frac{1}{2} \frac{y}{|y|^{3/2}}$ cannot be continuous on a rectangle of the form

$$[-b, b] \times [0, a].$$

4. Prove that $y(t) = -1$ is the only solution of the initial-value problem $\frac{d}{dt}y = t(y+1)$ and $y(0) = -1$.

It is easy to verify that $f(y,t) = t(y+1)$ and $\partial_y f(y,t) = t$ are continuous on any rectangle of the form

$$[-1-b, -1+b] \times [t_0, t_0+a]$$

By the theorem of uniqueness and existence, there is a unique solution for t on

$$[t_0, t_0+\alpha] \text{ where } \alpha = \min\left\{a, \frac{b}{M}\right\}$$

where $M = \max_{(y,t)\in R} |f(y,t)| = \max_{(y,t)\in R} |t(y+1)| = (t_0+a)\, b$,

that is, for $\alpha = \min\left\{a, \frac{1}{t_0+a}\right\} = \alpha_0$

this means that the problem

$$\begin{cases} \frac{d}{dt}y = t(y+1) & t > t_0 \\ y(t_0) = -1 \end{cases}$$

has always a single solution on $[t_0, t_0+\alpha_0]$.

As a consequence, $y(t) = -1$ is the only solution of

$$\begin{cases} \frac{dy}{dt} = t(y+1) & t > 0 \\ y(0) = -1 \end{cases}.$$