**MATH 4512 final exam:** Dec. 21, 2017 NAME:
(Each problem is 5 points)

1. Find the general solution of $\frac{d^3}{dt^3}y + y = 0$.
[One extra point if you verify that your answer satisfies the equation.]

Since the equation is of third order, there are three linearly independent solutions. Assuming that they are of the form $y(t) = e^{\lambda t}$, we see that we must have $\lambda^3 + 1^* = 0$. We then have that the general solution is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + C_3 e^{\lambda_3 t},$$

where $\lambda_1$, $\lambda_2$ and $\lambda_3$ are the three zeroes of $\lambda^3+1$, that is

$$\lambda_1 = -1$$
$$\lambda_2 = e^{\frac{\pi}{3}i} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$\lambda_3 = e^{-\frac{\pi}{3}i} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Equivalently,

$$y(t) = C_1 e^{\lambda_1 t} + D_1 \operatorname{Re}(e^{\lambda_2 t}) + D_2 \operatorname{Im}(e^{\lambda_2 t})$$
$$= C_1 e^{-t} + D_1 e^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + D_2 e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t.$$

* (Note also that $\lambda^3+1 = (\lambda+1)(\lambda^2-\lambda+1)$ ! )

2. Plot the phase portrait of the equation $\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$.

the characteristic polynomial of $A = \begin{bmatrix}0 & 1\\-1 & -2\end{bmatrix}$ is
$p(\lambda) := \det(A - \lambda Id) = \det\begin{bmatrix}-\lambda & 1\\-1 & -2-\lambda\end{bmatrix} = \lambda(2+\lambda)+1 = (\lambda+1)^2$.
thus there is only one eigenvalue, $\lambda = -1$. Let us find the corresponding eigenvector $\begin{bmatrix}a\\b\end{bmatrix}$:

$$\begin{bmatrix}1 & 1\\-1 & -1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}a\\b\end{bmatrix} = b\underbrace{\begin{bmatrix}-1\\1\end{bmatrix}}_{v_1}.$$ we take $b = 1$.

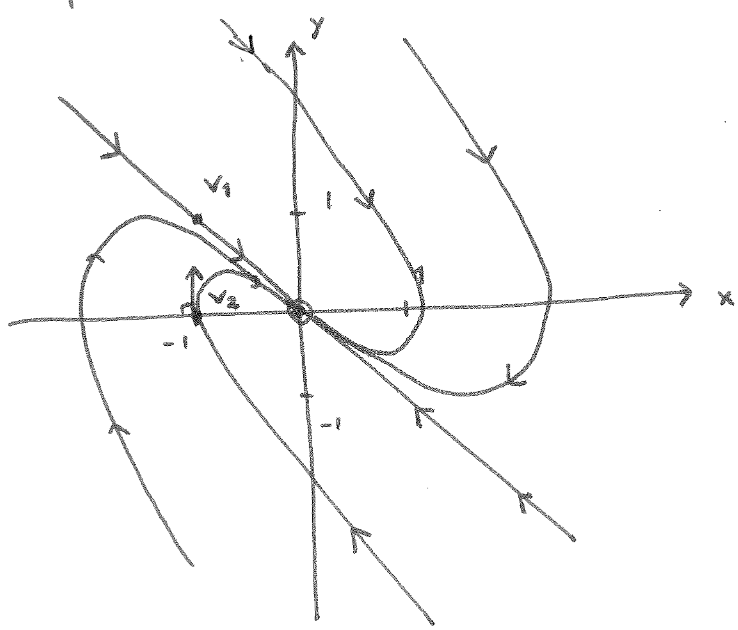
The generalized eigenvector $\begin{bmatrix}a\\b\end{bmatrix}$ is

$$\begin{bmatrix}1 & 1\\-1 & -1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix} \Rightarrow \begin{bmatrix}a\\b\end{bmatrix} = b\begin{bmatrix}-1\\1\end{bmatrix} + \underbrace{\begin{bmatrix}-1\\0\end{bmatrix}}_{v_2}.$$ We take $b=0$.

this implies that the solutions are

$$\begin{bmatrix}x\\y\end{bmatrix}(t) = c_1 e^{-t}\underbrace{\begin{bmatrix}-1\\1\end{bmatrix}}_{v_1} + c_2 e^{-t}\left(\underbrace{\begin{bmatrix}-1\\0\end{bmatrix}}_{v_2} + t\underbrace{\begin{bmatrix}-1\\1\end{bmatrix}}_{v_1}\right)$$

the phase portrait is then:




the equilibrium point $(0,0)$ is asymptotically stable.

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & 2\end{bmatrix}\underbrace{\begin{bmatrix}-1\\0\end{bmatrix}}_{v_2} = \begin{bmatrix}0\\1\end{bmatrix}$$

3. Find and plot the orbits of $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y\,(x^2+y^2) \\ -(x+3x^5)(x^2+y^2) \end{bmatrix}$.

The orbits are contained on the curves satisfying

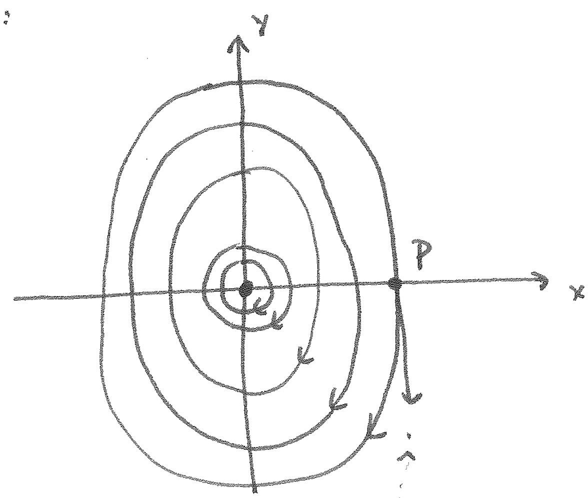
$$\frac{dy}{dx} = \frac{-(x+3x^5)(x^2+y^2)}{y\,(x^2+y^2)} = -\frac{(x+3x^5)}{y}$$

$$\Rightarrow \quad y\frac{dy}{dx} = -(x+3x^5)$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{1}{2}y^2\right) = \frac{d}{dx}\left(\frac{1}{2}x^2 + \frac{1}{2}x^6 + C\right)$$

$$\Rightarrow \quad y^2 + (x^2 + x^6) = C$$

When $|x|$ is small, $x^6$ is much smaller than $x^2$ and so the curve resembles a circle. The curve $y^2 + (x^2+x^6) = C$ lies inside the region $x^2+y^2 \leq C$. The only equilibrium point is $(0,0)$. All the other orbits are periodic. The phase space looks as follow:




$$\left.\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix}(t)\right|_{P=(a,0)} = \begin{bmatrix} 0 \\ -(a+3a^5)\,a^2 \end{bmatrix}$$

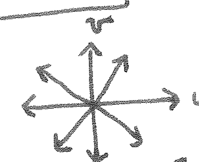
4. Sketch the phase portrait of $\frac{d}{dt}\begin{bmatrix} S \\ N \end{bmatrix} = \begin{bmatrix} S\,(1-(N+S)/2) \\ N\,(1-(N+S)) \end{bmatrix}$ for $S \geq 0$ and $N \geq 0$. Then, use the phase portrait to obtain the limit of $(S(t), N(t))$ as $t$ goes to infinity when $S(0) > 0$ and $N(0) > 0$.

the equilibrium points are these points $(S,N)$ such that $\frac{d}{dt}S = 0$ ($S=0$ or $N+S=2$) and $\frac{d}{dt}N = 0$ ($N=0$ or $N+S=1$). thus $(S,N) = (0,0), (0,1), (2,0)$.

Let us see the phase diagram of the linearized system around those equilibrium points. The equation is

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 1-\frac{N_0}{2}-S_0 & -\frac{S_0}{2} \\ -N_0 & 1-2N_0-S_0 \end{bmatrix}}_{A}\begin{bmatrix} u \\ v \end{bmatrix}$$

For $(S_0,N_0)=(0,0)$, $A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\Rightarrow$

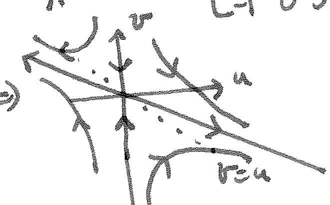



For $(S_0,N_0)=(0,1)$, $A=\begin{bmatrix} 1/2 & 0 \\ -1 & -1 \end{bmatrix}$ $\Rightarrow$ $\lambda = 1/2 \sim \begin{bmatrix} 0 & 0 \\ -1 & -3/2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = -1 \sim \begin{bmatrix} 3/2 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = b\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$

& $\begin{bmatrix} a \\ b \end{bmatrix} = b\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\Rightarrow$

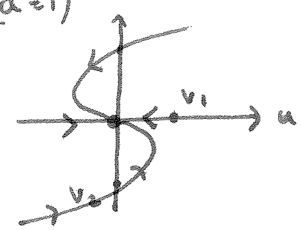



For $(S_0,N_0)=(2,0)$, $A=\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda=-1 \sim \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

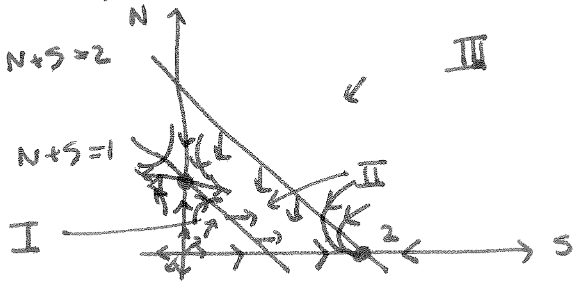
$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{v_1} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{v_2}$

(a=1) (a=0)

$\Rightarrow$




the phase portrait is then




Any orbit starting in III or I eventually gets into II.
Any orbit starting in II converges to (2,0) as t goes to infinity.