First MATH 4512 mid-term exam: Friday Sept. 29, 2017 NAME:

(Each problem is 5 points)

1. Find the solution of the initial-value problem $\frac{d}{dt}y = \frac{2t}{y(1+t^2)}$ and $y(2) = 3$.

Multiplying the equation by y we get

$$y \frac{d}{dt} y = \frac{2t}{(1+t^2)}$$

and so

$$\frac{d}{dt}\left(\frac{1}{2}y^2\right) = \frac{d}{dt} \ln(1+t^2).$$

Integrating from 2 to t, we get

$$\frac{1}{2}y^2(t) - \frac{1}{2}y^2(2) = \ln(1+t^2) - \ln(5).$$

Imposing the initial condition and rearranging we get

$$y^2(t) = 9 + 2\ln\left(\frac{1+t^2}{5}\right)$$

$$\Rightarrow y(t) = \pm\sqrt{9 + 2\ln\left(\frac{1+t^2}{5}\right)}$$

$$\Rightarrow y(t) = +\sqrt{9 + 2\ln\left(\frac{1+t^2}{5}\right)}$$ because $y(2) = +3$.

2. The population p of some species varies according to the equation

$$\frac{d}{dt}p = -p^3 + 3p^2 - 2p, t > 0.$$

Without finding the solution $p(t)$, what is the value of the population as the time t goes to infinity when the initial population is $p(0) = 3$ and when $p(0) = 1/3$?

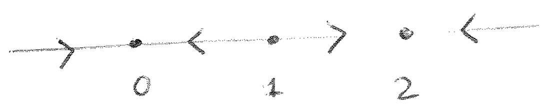
The equilibrium points of this equation are the solutions of

$$-p^3 + 3p^2 - 2p = 0$$

Since $-p^3 + 3p^2 - 2p = -p(p^2 - 3p + 2) = -p(p-2)(p-1)$ we see that the equilibrium points are

$$p = 0, \quad p = 1 \quad \text{and} \quad p = 2$$

The phase diagram is then :



So, if $p(0) = 3$, $p(t)$ goes to 2 as t goes to infinity.
If $p(0) = \frac{1}{3}$; $p(t)$ goes to 0 as t goes to infinity.

3. State the theorem of the existence and uniqueness of the solution of the initial-value problem $\frac{d}{dt}y = f(t, y), t > t_0$, and $y(t_0) = y_0$.

Suppose that f and $\frac{\partial}{\partial y} f$ are continuous on the rectangle $R = [t_0, t_0 + a] \times [y_0 - b, y_0 + b]$. then the initial value problem

$$\begin{cases} \frac{d}{dt} y = f(t, y) & t > t_0, \\ y(t_0) = y_0 \end{cases}$$

has a unique solution on the interval

$$t \in [t_0, t_0 + \alpha]$$

where $\alpha = \min\{a, \frac{b}{M}\}$, $M = \max_{(t,y) \in R} |f(t,y)|$.

4. Consider the initial-value problem $\frac{d}{dt}y = (t^2 + y^2)/2, t > 0$, and $y(0) = 0$. Show that the solution exists for $t \in [0, 1]$ and that $y(t) \in [-1, 1]$.

We apply the result stated in the previous problem. So, $f(t,y) = \frac{1}{2}(t^2+y^2)$ and $\frac{\partial}{\partial y} f(t,y) = y$ are clearly continuous in any rectangle

$$R = [0, a] \times [-b, b].$$

then

$$M = \max_{(t,y)\in R} |f(t,y)| = \frac{1}{2}(a^2+b^2)$$

and so, there is a unique solution for

$$t \in [0, \alpha]$$

where $\alpha = \min\{a, \frac{b}{M} = \frac{2b}{a^2+b^2}\}$.

the maximum of $\frac{2b}{a^2+b^2}$ occurs when $b=a$

$$\left(\text{since } 0 = \left(\frac{2b}{a^2+b^2}\right)' = \frac{2}{a^2+b^2} - \frac{4b^2}{(a^2+b^2)^2} = \frac{2a^2-2b^2}{(a^2+b^2)^2},\ a,b>0\right)$$

and so

$$\alpha = \min\{a, \tfrac{1}{a}\}.$$

the quantity $\min\{a, \frac{1}{a}\}$ is a maximum when $a = \frac{1}{a}$, that is when $a=1$. So, $\alpha = 1$. Since $b=a$, there is a unique solution for $t \in [0,\alpha] = [0,1]$. Finally, since

$$|Y(t) - Y(0)| = \left|\frac{t^2+y^2}{2}\right| \leq M = 1 \text{ in } R,$$

we get that $Y(t) \in [-1, +1]$ for $t \in [0,1]$.