Third MATH 4512 mid-term exam: Nov. 22, 2017 NAME:

(Each problem is 5 points)

1. The eigenvalues of a 3×3 matrix A are -1, 1, and 2. The respective eigenvectors are $\begin{bmatrix} 7 \\ -2 \\ 13 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find e^{At}.

This implies that we can write A as follows:

$$A = \begin{bmatrix} 7 & 1 & 1 \\ -2 & 0 & 1 \\ 13 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 7 & 1 & 1 \\ -2 & 0 & 1 \\ 13 & 1 & 1 \end{bmatrix}^{-1}$$

As a consequence, we have that

$$e^{At} = \begin{bmatrix} 7 & 1 & 1 \\ -2 & 0 & 1 \\ 13 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 15 & -6 & -9 \\ -2 & 6 & 2 \end{bmatrix} \frac{1}{6}$$

$$= \begin{bmatrix} 7e^{-t} & e^{t} & e^{2t} \\ -2e^{-t} & 0 & e^{2t} \\ 13e^{-t} & e^{t} & e^{2t} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 15 & -6 & -9 \\ -2 & 6 & 2 \end{bmatrix} \frac{1}{6}$$

$$= \begin{bmatrix} -7e^{-t} + 15e^{t} - 2e^{2t} & -6e^{t} + 6e^{2t} & 7e^{-t} - 9e^{t} + 2e^{2t} \\ 2e^{-t} - 2e^{2t} & 6e^{2t} & -2e^{-t} + 2e^{2t} \\ -13e^{-t} + 15e^{t} - 2e^{2t} & -6e^{t} + 6e^{2t} & 13e^{-t} - 9e^{t} + 2e^{2t} \end{bmatrix}$$

This implies that the following

$$X(t) = \begin{bmatrix} 7e^{-t} & e^{t} & e^{2t} \\ -2e^{-t} & 0 & e^{2t} \\ 13e^{-t} & e^{t} & e^{2t} \end{bmatrix}$$

is a fundamental matrix solution. Hence

$$e^{At} = X(t)\, X^{-1}(0)$$

$$= \begin{bmatrix} 7e^{-t} & e^{t} & e^{2t} \\ -2e^{-t} & 0 & e^{2t} \\ 13e^{-t} & e^{t} & e^{2t} \end{bmatrix} \begin{bmatrix} 7 & 1 & 1 \\ -2 & 0 & 1 \\ 13 & 1 & 1 \end{bmatrix}^{-1}$$

$$= (\dots).$$

2. Find e^{At} where $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

We have that $A = 2\,Id + N$, where

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$e^{At} = e^{2t} e^{Nt}$$
$$= e^{2t} \left\{ Id + tN + \frac{t^2}{2} N^2 + \frac{t^3}{6} N^3 \right\}$$

since $N^4 = 0$. Hence

$$e^{At} = e^{2t} \begin{bmatrix} 1 & t & t^2/2 & t^3/6 \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. Find e^{At} where $A = \begin{bmatrix} 0 & 4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

The characteristic polynomial is

$$p(\lambda) = \det \begin{bmatrix} -\lambda & 4 & 0 \\ -4 & -\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = \lambda^2 (3-\lambda) + 16 (3-\lambda)$$

$$= (3-\lambda)(\lambda^2+16).$$

So, the eigenvalues are $\lambda=3$, $\lambda = 4i$, $\lambda = -4i$

The eigenvector associated to $\lambda_1 = 3$ satisfies

$$\begin{bmatrix} -3 & 4 & 0 \\ -4 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v_3.$$ We take $v_3 = 1$.

The eigenvector associated to $\lambda_2 = 4i$ satisfies

$$\begin{bmatrix} -4i & 4 & 0 \\ -4 & -4i & 0 \\ 0 & 0 & 3-4i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix} v_2.$$ We take $v_2 = 1$.

This implies that the following are solutions of $\frac{dx}{dt} = Ax$:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{3t}, \quad \cos 4t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \sin 4t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \sin 4t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \cos 4t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Since they are linearly independent,

$$X(t) = \begin{bmatrix} 0 & \sin 4t & -\cos 4t \\ 0 & \cos 4t & \sin 4t \\ e^{3t} & 0 & 0 \end{bmatrix}$$

is a fundamental matrix solution. Then

$$e^{At} = X(t) X^{-1}(0)$$

$$= \begin{bmatrix} 0 & \sin 4t & -\cos 4t \\ 0 & \cos 4t & \sin 4t \\ e^{3t} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0 & \sin 4t & -\cos 4t \\ 0 & \cos 4t & \sin 4t \\ e^{3t} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos 4t & \sin 4t & 0 \\ -\sin 4t & \cos 4t & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}.$$

4. Find the solution y of $\frac{d^3}{dt^3}y+\frac{d^2}{dt^2}y+\frac{d}{dt}y+y=0$ such that $y(0)=0$, $\frac{d}{dt}y(0)=0$ and $\frac{d^2}{dt^2}y(0)=1$.

We can rewrite our equation in matrix form as follows:

$$\frac{d}{dt}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{dy}{dt} \\ \frac{d^2y}{dt^2} \\ \frac{d^3y}{dt^3} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -x_2-x_1-x_0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

then our solution is the first component of

$$e^{At}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let us find e^{At}. The characteristic polynomial is

$$p(\lambda) = \det\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -1 & -1 & -1-\lambda \end{bmatrix} = \lambda^2(-\lambda-1)-1-\lambda = (\lambda+1)(-\lambda^2-1).$$

So the eigenvalues are $\lambda=-1$, $\lambda=i$ and $\lambda=-i$.

The eigenvector associated to $\lambda_1=-1$ satisfies:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}. \text{ We take } v_2 = 1.$$

The eigenvector associated to $\lambda_2=i$ satisfies

$$\begin{bmatrix} -i & 1 & 0 \\ 0 & -i & 1 \\ -1 & -1 & -1-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2\begin{bmatrix} -i \\ 1 \\ i \end{bmatrix}. \text{ We take } v_2 = 1.$$

A fundamental solution is

$$X(t) = \begin{bmatrix} -e^{-t} & \sin t & -\cos t \\ e^{-t} & \cos t & \sin t \\ -e^{-t} & -\sin t & \cos t \end{bmatrix}$$

Since $X(0) = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and $X(0)^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}\frac{1}{2}$,

we have that

$$e^{At} = \begin{bmatrix} -e^{-t} & \sin t & -\cos t \\ e^{-t} & \cos t & \sin t \\ -e^{-t} & -\sin t & \cos t \end{bmatrix}\begin{bmatrix} -1 & 0 & -1 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}\frac{1}{2}$$

So,

$$e^{At}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -e^{-t} & \sin t & -\cos t \\ e^{-t} & \cos t & \sin t \\ -e^{-t} & -\sin t & \cos t \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ +1 \end{bmatrix}\frac{1}{2}$$

and

$$y(t) = \frac{1}{2}\left(+e^{-t} + \sin t - \cos t\right).$$