Homework #3: Finite element methods. Due on Monday, March 25.

In what follows, u is the solution of the following boundary-value problem:

$$-u'' = f \text{ in } (0,1), \qquad u = u_D \text{ on } \{0,1\}.$$

In the first five problems, take $f = 1$ and $u_D = 0$. In the last, which is optional, take f and u_D in such a way that the exact solution is

$$u(x) = tan^{-1}(100(.5 - x)).$$

Note that the slope of u is very big close to $x = 1/2$.

1. (4 pts.) Define the approximation u_h given by the continuous Galerkin (CG) finite element method using piecewise linear approximations; use uniform partitions of the domain $(0,1)$. Let N be the number of intervals of the partition and set $h := 1/N$. Find the local matrices. Then show how to assemble the global matrices. For $N = 5$, display the matrix equation defining the approximate solution.

2. (4 pts.) To explore the convergence properties of the CG method, fill the table below. Let e_h be either $\|u - u_h\|_{L^2(0,1)}$ or $\|u' - u_h'\|_{L^2(0,1)}$, with $h = 1/N$. Assuming that $e_h = C\,h^\alpha$, we have that $\alpha = -\ln(e_h/e_{2h})/\ln 2$. Motivated by this, set

$$\alpha_h := -\ln(e_h/e_{2h})/\ln 2, \quad C_h := e_h\,h^{-\alpha_h}.$$

TABLE 0.4
History of convergence of the approximation given by the continuous Galerkin method

N	$\|u - u_h\|_{L^2(0,1)}$	α_h	C_h	$\|u' - u_h'\|_{L^2(0,1)}$	α_h	C_h
2		-	-		-	-
4						
8						
...						

Was the assumption $e_h = C\,h^\alpha$ reasonable? Do your results agree with the theory?

3. (4 pts.) Let $\widetilde{u}_h$ be the piecewise linear function such that $\widetilde{u}_h(x_i) = u(x_i)$ for all the partition points $x_i, i = 0, \ldots, N$. Find the kernel K_i such that, for any $x \in (x_{i-1}, x_i)$,

$$u(x) - \widetilde{u}(x) = \int_{x_{i-1}}^{x_i} K_i(x,s)\,u''(s)\,ds.$$

For uniform partitions, compute analytically $\|u - \widetilde{u}_h\|_{L^2(0,1)}$ and $\|u' - \widetilde{u}_h'\|_{L^2(0,1)}$, and compare it with $\|u - u_h\|_{L^2(0,1)}$ and $\|u' - u_h'\|_{L^2(0,1)}$, respectively.

4. (4 pts.) Prove that $\widetilde{u}_h = u_h$.

5. (4 pts.) Find a functional Φ such that $\|u' - u_h'\|_{L^2(0,1)} \leq \Phi(u_h)$. Test the performance of your error estimate by computing the so-called computational effectivity index

$$EI_h := \Phi(u_h)/\|u' - u_h'\|_{L^2(0,1)}.$$

Fill the table above; use uniform partitions. Is the actual error close to $\Phi(u_h)$?

TABLE 0.5
Performance of the effectivity index for the CG method

N	EI_h
2	
4	
8	
...	

6. (8 pts.)(This exercise is optional) Given any tolerance τ, devise a method to find a partition $\{x_i\}_{i=0}^N$ such that

$$\Phi(u_h) \leq \tau.$$

Evaluate the performance of your method by completing the following table and by plotting u and its approximation u_h.

TABLE 0.6
Performance of the adaptive method for the CG method

τ	EI_h	Φ/τ
10^{-1}		
10^{-2}		
10^{-3}		
...		