Homework #9: Interpolation with polynomials. Due on Monday, November 5.

Each question is four points. The last two problems are *optional*.

In what follows, $P_N u$ is the polynomial interpolant of the function $u : (-1, 1) \mapsto \mathbb{R}$ at the points x_j, $j = 0, \ldots, N$.

1. **Equidistant points: Runge's phenomenon**. Consider the function $u(x) = |x|$ and the interpolation points $x_j := -1 + \frac{2}{N} j$, $j = 0, \ldots, N$. Plot the interpolant $P_N u$ for various values of N. What happens when N increases. Do we have convergence?

2. **Chebyshev points**. Repeat the above exercise with the Chebyshev points $x_j := \cos \pi \frac{2j+1}{2N+2}\, j = 0, \ldots, N$. Plot the interpolant $P_N u$ for various values of N. What happens when N increases. Do we have convergence? What kind of convergence?

The **Lebesgue**'s constant of the operator $P_n u(x) = \sum_{j=0}^{N} u(x_j)\, \ell_j(x)$ is

$$\Lambda_N := \max_{-1 \le x \le 1} \sum_{j=0}^{N} |\ell_j(x)|$$

It has been proven that we always have the following lower bound:

$$\Lambda_N > \frac{2}{\pi} \ln N + \big[\frac{2}{\pi} \ln(8/\pi + \gamma)\big] \approx \frac{2}{\pi} \ln N + 0.96,$$

where γ is Euler's constant. Next, we explore the actual behavior of the Lebesgue constant for the cases we have considered in the previous problems.

3. For the case of equidistant points, plot the ratio $\Lambda_N / (2^{N+1}/(e\, N \ln N))$ for many values of N. To what value does this ratio seem to go as N goes to infinity? Is there any relation between this behavior and the results of problem number 1?

4. For the case of Chebyshev points, plot the ratio $\Lambda_N / (2 \ln N / \pi)$ for many values of N. To what value does this ratio seem to go as N goes to infinity? Is this behavior consistent with the results of problem number 1? Why?

5. Consider the error $e_N(x) := u(x) - P_N u(x)$ where $x \in (-1, 1)$. For each $m \in \{0, 1, \ldots, N\}$, find the kernel $K_{m,N}(x, s)$ such that

$$e_N(x) = \int_{-1}^{1} K_{m,N}(x, s)\, u^{(m+1)}(s)\, ds.$$

6. Take $m = N$ in the previous problem and deduce Cauchy's error estimate.