Homework #3. A solution

1. Using Taylor expansion at (t^n, x_j), we get

$$u(t^{n+1}, x_j) = \left(u + u_t \Delta t + \frac{u_{tt}}{2}(\Delta t)^2 + \frac{u_{ttt}}{6}(\Delta t)^3\right)\Big|_{(t^n, x_j)} + O((\Delta t)^4)$$

$$u(t^n, x_{j+1}) = \left(u + u_x \Delta x + \frac{u_{xx}}{2}(\Delta x)^2 + \frac{u_{xxx}}{6}(\Delta x)^3\right)\Big|_{(t^n, x_j)} + O((\Delta x)^4)$$

$$u(t^n, x_{j-1}) = \left(u - u_x \Delta x + \frac{u_{xx}}{2}(\Delta x)^2 - \frac{u_{xxx}}{6}(\Delta x)^3\right)\Big|_{(t^n, x_j)} + O((\Delta x)^4)$$

Therefore, we obtain

$$\frac{1}{\Delta t}\left(u(t^{n+1}, x_j) - u(t^n, x_j)\right) = \left(u_t + \frac{u_{tt}}{2}\Delta t + \frac{u_{ttt}}{6}(\Delta t)^2\right)\Big|_{(t^n, x_j)} + O((\Delta t)^3)$$

$$\frac{a}{\Delta x}\left(\hat{u}_{j+\frac{1}{2}} - \hat{u}_{j-\frac{1}{2}}\right) = \frac{a}{\Delta x}\left[\frac{1}{2}\left(u(t^n, x_{j+1}) - u(t^n, x_{j-1})\right) - \frac{\alpha}{2}\left(u(t^n, x_{j+1}) + u(t^n, x_{j-1}) - 2u(t^n, x_j)\right)\right]$$

$$= \frac{a}{\Delta x}\left[u_x \Delta x + \frac{u_{xxx}}{6}(\Delta x)^3 - \alpha \frac{u_{xx}}{2}(\Delta x)^2 + O((\Delta x)^4)\right]\Big|_{(t^n, x_j)}$$

$$= \left[a u_x - \frac{\alpha a}{2} u_{xx} \Delta x + \frac{a u_{xxx}}{6}(\Delta x)^2\right]\Big|_{(t^n, x_j)} + O((\Delta x)^3)$$

Note that $(u_t + a u_x)\big|_{(t^n, x_j)} = 0$.

Then $\begin{cases} u_{tt} + a u_{xt} = 0, \\ u_{tx} + a u_{xx} = 0 \end{cases}$

If u is smooth enough, we have

$$u_{tt} = a^2 u_{xx}$$

Similarly, we have $\begin{cases} u_{ttx} + a u_{xxt} = 0 \\ u_{ttt} + a u_{xtt} = 0 \\ u_{txx} + a u_{xxx} = 0 \end{cases}$

Then we get $u_{ttt} = -a^3 u_{xxx}$.

We also know $\Delta t = \frac{\nu}{a}\Delta x$, where ν is an constant.

Hence, the local truncation error is

$$LTE_j^n = \frac{1}{\Delta t}\left(u(t^{n+1}, x_j) - u(t^n, x_j)\right) + \frac{a}{\Delta x}\left(\hat{u}^n_{j+\frac{1}{2}} - \hat{u}^n_{j-\frac{1}{2}}\right)$$

$$= \left[u_t + \frac{u_{tt}}{2}\Delta t + \frac{u_{ttt}}{6}(\Delta t)^2 + a u_x - \frac{\alpha a}{2} u_{xx}\Delta x + \frac{a u_{xxx}}{6}(\Delta x)^2\right]\Big|_{(t^n, x_j)} + O((\Delta t)^3 + (\Delta x)^3)$$

$$= \left[u_t + a u_x + \frac{a^2 u_{xx}}{2}\frac{\nu}{a}\Delta x - \frac{\alpha a}{2} u_{xx}\Delta x - \frac{a^3}{6} u_{xxx}\frac{\nu^2}{a^2}\Delta x^2 + \frac{a u_{xxx}}{6}\Delta x^2\right]\Big|_{(t^n, x_j)} + O((\Delta x)^3)$$

$$= \left[a(\nu - \alpha)\frac{u_{xx}}{2}\Delta x + a(1 - \nu^2)\frac{u_{xxx}}{6}(\Delta x)^2\right]\Big|_{(t^n, x_j)} + O(\Delta x^3)$$

where $\nu = \frac{a \Delta t}{\Delta x} > 0$.

From the LTE_j^n, we can clearly see that

① If $\alpha \neq \nu$, the scheme is first order, namely $O(\Delta x)$

② If $\alpha = \nu$ and $\nu \neq 1$, the scheme is second order, namely $O(\Delta x^2)$

③ If $\alpha = \nu$ and $\nu = 1$, the scheme is third order, namely $O(\Delta x^3)$
(in fact, the scheme is exact in this case.)

- For upwinding, downwing and centered schemes, we take α to be constants ($\alpha = 1, \alpha = -1, \alpha = 0$). Generally, we expect to get first order (unless $\alpha = \nu$ by chance).
- For Lax-Friedrich, we take $\alpha = \frac{1}{\nu}$. So we expect to get first order (unless $\nu = 1$, which will be third order).
- For Lax-Wendroff, we take $\alpha = \nu$. So we expect to get second order accuracy (unless $\nu = 1$, which will be third order).

2. Using discrete Fourier transform, we have

$$0 = \frac{1}{\Delta t}(\tilde{u}_k^{n+1} - \tilde{u}_k^n) + \frac{a}{\Delta x}\left[\frac{1}{2}(\tilde{u}_k^n e^{i\Delta x k} + \tilde{u}_k^n) - \frac{\alpha}{2}(\tilde{u}_k^n e^{i\Delta x k} - \tilde{u}_k^n) - \frac{1}{2}(\tilde{u}_k^n + \tilde{u}_k^n e^{-i\Delta x k}) + \frac{\alpha}{2}(\tilde{u}_k^n - \tilde{u}_k^n e^{-i\Delta x k})\right]$$

This is

$$0 = \frac{1}{\Delta t}(\tilde{u}_k^{n+1} - \tilde{u}_k^n) + \frac{a}{\Delta x}\left[\frac{1}{2}(e^{i\Delta x k} - e^{-i\Delta x k}) - \frac{\alpha}{2}(e^{i\Delta x k} + e^{-i\Delta x k} - 2)\right]\tilde{u}_k^n$$

Therefore,

$$\tilde{u}_k^{n+1} = \left(1 - \nu[i\sin\theta + \alpha(1-\cos\theta)]\right)\tilde{u}_k^n$$
$$= (1 - \alpha\nu(1-\cos\theta) - i\nu\sin\theta)\tilde{u}_k^n$$

where $\theta = \Delta x k$.

Then

$$\tilde{u}_k^{n+1} = g(\nu, \theta)\tilde{u}_k^n$$

where $g(\nu, \theta) = 1 - \alpha\nu(1-\cos\theta) - i\nu\sin\theta$

To make the scheme L^2- stable, we need $|g(\nu,\theta)| \leq 1$ for any $\theta \in [-\pi,\pi]$

Since

$$|g(\nu,\theta)| = [1-\alpha\nu(1-\cos\theta)]^2 + \nu^2\sin^2\theta$$

$$= [1-\alpha\nu+\alpha\nu\cos\theta]^2 + \nu^2(1-\cos^2\theta)$$

$$= (1-\alpha\nu)^2 + \alpha^2\nu^2\cos^2\theta + 2\alpha\nu(1-\alpha\nu)\cos\theta + \nu^2 - \nu^2\cos^2\theta$$

$$= \nu^2(\alpha^2-1)\cos^2\theta + 2\alpha\nu(1-\alpha\nu)\cos\theta + (1-\alpha\nu)^2+\nu^2$$

There is a quadratic function of $\cos\theta \in [-1, 1]$.

Hence

① If $\alpha^2-1<0$, we know the maximum of $|g(\nu,\theta)|$ is at $\cos\theta = -\dfrac{\alpha\nu(1-\alpha\nu)}{\nu^2(\alpha^2-1)} \in [-1,1]$

② If $\alpha^2-1\geq 0$, the maximum of $|g(\nu,\theta)|$ is at $\cos\theta = 1$ or $\cos\theta = -1$.

Lets us first consider case ②.

If $\cos\theta=1$, $|g(\nu,\theta)| = 1$

If $\cos\theta=-1$, $|g(\nu,\theta)| = (1-2\alpha\nu)^2$

We need $|1-2\alpha\nu| \leq 1$

$$\Rightarrow \quad -1 \leq 2\alpha\nu - 1 \leq 1$$

$$0 \leq 2\alpha\nu \leq 2$$

$$\Rightarrow \quad 0 \leq \alpha\nu \leq 1$$

We also know $\alpha^2 \geq 1$

Combining these two inequalities, we have

$$\alpha \geq 1, \ \nu \leq \frac{1}{\alpha} \leq 1$$

Now let's consider case ①

if $\alpha^2-1<0$ and $\cos\theta = -\frac{\alpha\nu(1-\alpha\nu)}{\nu^2(\alpha^2-1)} \in [-1,1]$, the maximum of $|g(\nu,\theta)|$ is

$$-\frac{[\alpha\nu(1-\alpha\nu)]^2}{\nu^2(\alpha^2-1)} + (1-\alpha\nu)^2+\nu^2$$

$$= -\frac{\alpha^2(1-\alpha\nu)^2}{\alpha^2-1} + (1-\alpha\nu)^2+\nu^2$$

$$= -\frac{(1-\alpha\nu)^2}{\alpha^2-1} + \nu^2$$

$$= \frac{\nu^2-2\alpha\nu+1}{1-\alpha^2}$$

$$= \frac{(\nu-\alpha)^2}{1-\alpha^2}+1$$

Since $\alpha^2<1$ and we need $|g(\nu,\theta)|\le 1$

The only possible case is $\nu=\alpha$ and $\cos\theta=1$

So if $0<\nu=\alpha<1$, we have L^2-stability.

Next,

If $\alpha^2<1$ and $\cos\theta = -\frac{\alpha\nu(1-\alpha\nu)}{\nu^2(\alpha^2-1)} \notin [-1,1]$,

then the maximum of $|g(\nu,\theta)|$ is at $\cos\theta=1$ or $\cos\theta=-1$.

From previsous disscussion, we know this requires

$0\le \alpha\nu \le 1$

Notice that if $0 \le \alpha\nu \le 1$, we know $\cos\theta = \dfrac{\alpha\nu(1-\alpha\nu)}{\nu^2(1-\alpha^2)} > 0$

So we need $\dfrac{\alpha\nu(1-\alpha\nu)}{\nu^2(1-\alpha^2)} > 1$

$\Rightarrow \quad \alpha\nu - \alpha^2\nu^2 > \nu^2 - \nu^2\alpha^2$

$\Rightarrow \quad \alpha\nu > \nu^2$

$\Rightarrow \quad \alpha > \nu$

Now combine these requirements together to get

$$\begin{cases} \alpha^2 < 1 \\ 0 \le \alpha\nu \le 1 \\ 0 < \nu < \alpha \end{cases} \Rightarrow \begin{cases} 0 < \alpha < 1 \\ 0 < \nu\alpha \le 1 \\ 0 < \nu < \alpha \end{cases} \Rightarrow \begin{cases} 0 < \alpha < 1 \\ 0 < \nu < \alpha \end{cases}$$

Finally, we sum up all cases for which we have L^2-stability:

1. $0 < \nu = \alpha < 1$
2. $0 < \alpha < 1$ and $0 < \nu < \alpha$
3. $\alpha \ge 1$, and $\nu \le \frac{1}{\alpha}$

Case 1 and 2 lead to $0 < \nu \le \alpha < 1$.

So the final result is

if $(0 < \nu \le \alpha < 1)$ or $(\alpha \ge 1$ and $\nu \le \frac{1}{\alpha} \le 1)$, we have L^2- stability.

For the downwinding scheme and centered scheme $\alpha=-1$ and $\alpha=0$, they are unstable.

For the upwinding scheme $\alpha=1$, we need $\nu \leq 1$ to get a stable scheme.

For the Lax-Friedrich scheme, we have $\alpha=\frac{1}{\nu}$ (or $\nu=\frac{1}{\alpha}$). From the previous results we need $\nu \leq 1$ to get a stable scheme.

For the Lax-Wendroff scheme, we have $\nu=\alpha$. If $\nu \leq 1$, we have a stable scheme.

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3. If $T=2\pi$ and $a=1$, then the problem is

$$\begin{cases} u_t+u_x=0 & \text{in } (0,2\pi)\times(0,2\pi) \\ u(x,0)=u_0(x) & \forall\ x\in(0,2\pi) \\ u(x+2\pi,t)=u(x,t) & \end{cases}$$

It's easy to see that $C=x-t$ is a charactistic line for any constant C.

Hence the solution of this problem is

$$u(x,t) = u(c,0)$$
$$= u_0(c') \quad \text{where } c' \equiv x-t \pmod{2\pi}$$

So $u(x,T) = u(x,2\pi)$
$$= u_0(x-2\pi)$$
$$= u_0(x) \quad \text{(periodic boundary).}$$

Now for the centered method $\alpha=0$, we consider two cases ① $u_0(x) = \sin x$ (continuous)

② $u_0(x) = \chi_{[0.8\pi,\,1.2\pi]}(x)$ (discontinuous)

We compute $\{u_j^N\}_{j=0}^{M-1}$ at $t=T$ and calculate the error

$$\|u(T) - u_h(T)\|_h^2 = \sum_{j=0}^{M-1} \left(\frac{1}{2}(u(x_j^-,T) + u(x_j^+,T)) - u_j^N\right)^2 \Delta x$$

The numerical results are as follows ($\nu=0.5$)

mesh_list	error_list	order_list	Ch_list
4	5.8993	0	0
8	3.6239	0.70298	2.488
16	1.4842	1.2879	8.3957
32	0.63948	1.2147	6.8547
64	0.29555	1.1135	4.8263
128	0.14208	1.0567	3.8106
256	4.877e+08	-31.677	4.0281e-69
512	2.2798e+33	-81.951	3.4037e-190
1024	7.36e+82	-164.47	0
2048	1.1553e+182	-329.52	0

Centered scheme: smooth case

mesh_list	error_list	order_list	Ch_list
4	2.9496	0	0
8	2.7975	0.076401	0.5219
16	12.336	-2.1407	0.0051926
32	240.26	-4.2836	1.3645e-05
64	1.6508e+05	-9.4243	2.4972e-13
128	1.5809e+11	-19.869	3.4054e-32
256	2.3912e+23	-40.46	1.3892e-75
512	9.1408e+47	-81.661	8.3462e-175
1024	2.2531e+97	-164.08	0
2048	2.305e+196	-328.9	0

Centered scheme: discontinuous case

We can see that for both cases, the numerical results of the centered scheme blow up when M gets large enough. This matches with the theory that the centered scheme is unstable.

4. Based on problem 3, we know

if $u_0 = \sin x$,

$$u(x, 2\pi) = \sin x$$

if $u_0 = \chi_{[0.8\pi, 1.2\pi]}(x)$,

$$u(x, 1) = \chi_{[0.8\pi, 1.2\pi]}(c) \quad \text{where } c \equiv x - 1 \pmod{2\pi}$$

We consider two methods: upwinding ($\alpha = 1$) and Lax-Wendroff ($\alpha = \nu$).

The numerical results are as following ($\nu = 0.5$)

mesh_list	error_list	order_list	Ch_list
4	1.6617	0	0
8	1.2731	0.38431	0.45055
16	0.81979	0.63501	0.75884
32	0.47105	0.79937	1.1969
64	0.2534	0.89449	1.6643
128	0.13154	0.94585	2.0606
256	0.067035	0.97256	2.3458
512	0.03384	0.98619	2.5299
1024	0.017001	0.99307	2.6409
2048	0.0085212	0.99653	2.7049
4096	0.0042657	0.99826	2.7409
8192	0.0021341	0.99913	2.7608
16384	0.0010674	0.99957	2.7716
32768	0.00053378	0.99978	2.7775

Upwinding scheme: continuous case

mesh_list	error_list	order_list	Ch_list
4	0.83232	0	0
8	0.80513	0.047919	0.14157
16	0.51641	0.6407	0.48561
32	0.37485	0.4622	0.29605
64	0.32016	0.22751	0.13126
128	0.27508	0.21895	0.12666
256	0.22758	0.27349	0.16504
512	0.19203	0.24506	0.14097
1024	0.16077	0.25632	0.15122
2048	0.13525	0.24938	0.14412
4096	0.11375	0.24968	0.14445
8192	0.095705	0.24926	0.14395
16384	0.080459	0.25034	0.14536
32768	0.067649	0.25019	0.14515

Upwinding scheme: discontinuous case

mesh_list	error_list	order_list	Ch_list
4	1.9394	0	0
8	0.77514	1.3231	1.9322
16	0.21097	1.8775	6.1194
32	0.053496	1.9795	8.1209
64	0.013408	1.9963	8.6072
128	0.0033538	1.9993	8.7137
256	0.00083855	1.9998	8.7384
512	0.00020964	2	8.7445
1024	5.2411e-05	2	8.7461
2048	1.3103e-05	2	8.7465
4096	3.2757e-06	2	8.7467
8192	8.1893e-07	2	8.7465
16384	2.0473e-07	2	8.7496
32768	5.1171e-08	2.0003	8.7712

Lax-Wendroff scheme: continuous case

mesh_list	error_list	order_list	Ch_list
4	1.124	0	0
8	0.84593	0.41009	0.31586
16	0.46716	0.85662	0.79939
32	0.33436	0.48252	0.28334
64	0.26959	0.31062	0.15616
128	0.25134	0.10115	0.065345
256	0.2009	0.32318	0.1919
512	0.15695	0.35614	0.23038
1024	0.12515	0.3266	0.19161
2048	0.10119	0.30662	0.16683
4096	0.081771	0.30743	0.16787
8192	0.064368	0.34526	0.22994
16384	0.052366	0.2977	0.1498
32768	0.041904	0.32154	0.18879

Lax-Wendroff scheme: discontinuous case

From the numerical results above, we can clearly see that upwinding scheme is 1st order and the Lax-Wendroff scheme is 2nd order for the smooth case. This matches with our theory that both schemes are stable and are 1st and 2nd order of accuracy, respectively.

(the optimal problem:)

For the discontinuous case, the results of both schemes seem to converge. The order of accuracy for upwinding scheme is very close to 0.25 and the order of accuracy for Lax-Wendroff is close to $\frac{1}{3}$.

Optional:

Let U_h be the numerical solution. Then based on the class notes, we know

$$\|u(T) - U_h(T)\|_{L^2} \le 2\|u_0\|_{L^2}$$

$$\|u(T) - U_h(T)\|_{L^2} \le C\frac{Ta}{\nu}(\Delta x)^{\beta}\left\|\frac{d^{\beta+1}u_0}{dx^{\beta+1}}\right\|_{L^2} \quad (\beta \text{ is the order of the scheme})$$

where u_0 is smooth enough.

If u_0 is not smooth enough, recall the convergence theorem 2.5 we learnt last semester, we have

$$\|u(T) - u_k(T)\| \leq 2^{1-\theta}\left(C\frac{Ta}{\nu}(\Delta x)^{\beta}\right)^{\theta} |u_0|_{\theta}$$

If $u_0 = \chi_{[0.8\pi, 1.2\pi]}(x)$, based on the example we did last semester, we know

$$\frac{1}{2} = \theta(\beta+1)$$

So $\theta = \frac{1}{2(\beta+1)}$

Then $\|u(T) - u_k(T)\| \leq C(\Delta x)^{\frac{\beta}{2(\beta+1)}} |u_0|_{\theta}$

For upwinding scheme, we know $\beta = 1$. Then

$$\|u(T) - u_k(T)\| \leq C(\Delta x)^{\frac{1}{4}} |u_0|_{\theta}$$

So the order of accuracy is 0.25.

For Lax-Wendroff scheme, we know $\beta = 2$. Then

$$\|u(T) - u_k(T)\| \leq C(\Delta x)^{\frac{1}{3}} |u_0|_{\theta}$$

So the order of accuracy is $\frac{1}{3}$.