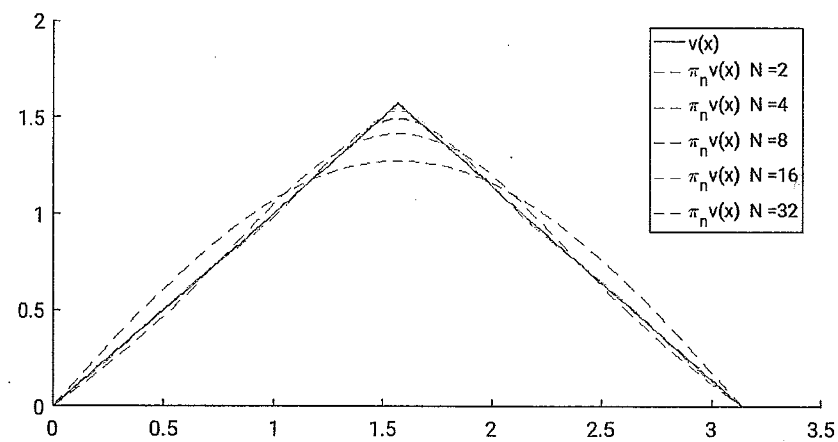
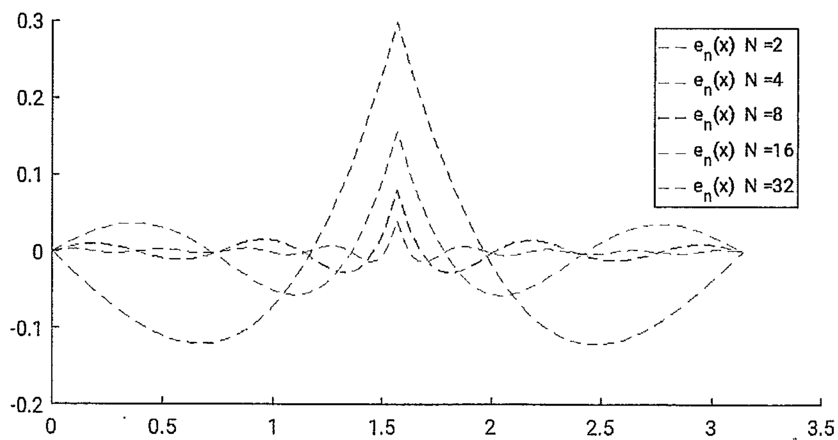
Homework #5.

1. In the figures below, we plot $\Pi_n v(x)$ and $e_n(x) := v(x) - \Pi_n v(x)$ for a few values of n. We see how $\Pi_n v$ converges to v as n increases and how it oscillates around v more and more as n increases.





In the history of convergence table below, we see that the assumption $e_n = C n^{-\alpha}$ is reasonable as α_n seems to be converging to 1.5 and C_n to a constant close to 0.65.

n	e_n	α_n	C_n
2	1.9 e-1	-	-
4	7.7 e-2	1.33	0.49
8	2.8 e-2	1.44	0.57
16	1.0 e -2	1.48	0.62
32	3.6 e-3	1.50	0.64
64	1.3 e-3	1.50	0.65

Table 1: History of convergence in L_2 for $v(x) = \pi/2 - |x - \pi/2|$

Let us show that $\alpha = 1.5$ is the theoretical order of convergence. We know that

(*) $$\|v - \Pi_n v\|_{L^2(0,\pi)} \leq K_{n+1}^{s\theta} \|v\|_{s,\theta}$$

For $s=1$ and $\theta=1$, we get

$$\|v - \Pi_n v\|_{L^2(0,\pi)} \leq K_{n+1} \left\|\frac{d^2}{dx^2} v\right\|_{L^2(0,\pi)} = (n+1)^{-2} \left\|\frac{d^2 v}{dx^2}\right\|_{L^2(0,\pi)},$$

provided $v \in \mathcal{C}^2[0,\pi]$ and $v=0$ at $x \in \{0,\pi\}$. So, we can use the estimate (*) with $s=1$ and some θ we have to find.

To do that, we begin by noting that

$$\|v\|_{1,\theta} = \sup_{t>0} t^{-\theta} K(v,t)$$

$$K(v,t) \leq \inf_{\substack{\epsilon>0 \\ \varphi_\epsilon \in \mathcal{C}^2[0,\pi] \\ \varphi_\epsilon(0)=\varphi_\epsilon(\pi)=0}} \left(\|v-\varphi_\epsilon\|_{L^2(0,\pi)} + t \left\|\frac{d^2\varphi_\epsilon}{dx^2}\right\|_{L^2(0,\pi)} \right)$$

We pick

$$\varphi_\epsilon(x) = v(x) + \epsilon\, p\left(\frac{x-[\frac{\pi}{2}-\epsilon]}{\epsilon}\right) \qquad x \in (0, \tfrac{\pi}{2}),$$

and

$$\varphi_\epsilon(x) = \varphi(\pi - x) \qquad x \in (\tfrac{\pi}{2}, \pi).$$

We take $p \in \mathcal{C}^2[0,1]$ such that

$$p(0) = p'(0) = p''(0) = 0,$$
$$p'(1) = -v'(\tfrac{\pi}{2}).$$

then

$$\|\varphi_\epsilon - v\|^2_{L^2(0,\pi)} = 2\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \left[\epsilon\, p\left(\frac{x-[\frac{\pi}{2}-\epsilon]}{\epsilon}\right)\right]^2 dx$$

$$= \left[2\int_0^1 p^2(\hat{x}) \cdot d\hat{x}\right] \epsilon^3 =: c_0^2\, \epsilon^3$$

$$\left\|\frac{d^2}{dx^2}\varphi_\epsilon\right\|^2_{L^2(0,\pi)} = 2\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \left[\frac{1}{\epsilon} p''\left(\frac{x-[\frac{\pi}{2}-\epsilon]}{\epsilon}\right)\right]^2 dx$$

$$= \left[2\int_0^1 p''^2(\hat{x})\, d\hat{x}\right] \epsilon^{-1} =: c_1^2\, \epsilon^{-1}$$

$$\Rightarrow \quad K(v,t) \leq c_0\, \epsilon^{3/2} + t\, c_1\, \epsilon^{-1/2}$$

If we take $\sigma = t^{1/2}$, we get

$$K(v,t) \leq (c_0 + c_1)\, t^{3/4}$$

Since $\sigma \in [0, \pi/2]$, this holds for $t \in [0, \sqrt{\pi/2}\,]$.
For $t > \sqrt{\frac{\pi}{2}}$, we use the fact that

$$K(v,t) \leq \|v\| .$$

then

$$\|v\|_{1,\theta} \leq \max\left\{ \sup_{0 < t \leq \sqrt{\pi/2}} t^{-\theta} K(v,t),\ \sup_{t > \sqrt{\frac{\pi}{2}}} t^{-\theta} K(v,t) \right\}$$

$$= \max\left\{ (c_0 + c_1) \sup_{0 < t \leq \sqrt{\frac{\pi}{2}}} t^{-\theta + \frac{3}{4}},\ \|v\| \left(\sqrt{\tfrac{\pi}{2}}\right)^{-\theta} \right\}$$

and we see that the maximum value of θ which renders the right-hand side bounded is $\theta = 3/4$.
Hence

$$\|v - \Pi_n v\| \leq K_{n+1}^{1 \cdot \frac{3}{4}} \|v\|_{1,\frac{3}{4}}$$

$$\leq (n+1)^{-\frac{3}{2}} \|v\|_{1,\frac{3}{4}} ,$$

and we see that $\alpha = \frac{3}{2}$.

2. Below is the history of convergence of the orthogonal projection in the L^∞-norm. Our assumption that $e_n = C n^{-\alpha}$ is reasonable because α_n seems to be converging to 1 and C_n to some constant close to 0.64.

n	e_n	α_n	C_n
2	3.0 e-1	-	-
4	1.6 e-1	0.93	0.57
8	8.0 e-2	0.98	0.61
16	4.0 e-2	0.99	0.63
32	2.0 e-2	1.00	0.63
64	1.0 e-2	1.00	0.64

Table 2: History of convergence in L_∞ for $v(x) = \pi/2 - |x - \pi/2|$

Let us justify this theoretically.
We know that

$$|\varphi(x) - \Pi_n \varphi(x)| \leq \sqrt{\frac{2}{\pi}} \frac{n^{-2s+1/2}}{\sqrt{4s+1}} \|T^{-s}\varphi\|_{L^2(0,\pi)} \qquad s > s_0 = \frac{1}{4}$$

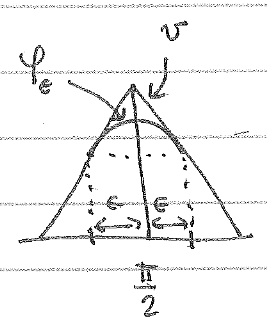
For $\varphi \in C^2[0,\pi]$ such that $\varphi(0) = \varphi(\pi) = 0$, we have

$(s = 1/2)$ (i) $\displaystyle |\varphi(x) - \Pi_n \varphi(x)| \leq \sqrt{\frac{2}{\pi}} \frac{n^{-1/2}}{\sqrt{3}} \left\|\frac{d}{dx}\varphi\right\|_{L^2(0,\pi)}$

$(s = 1)$ (ii) $\displaystyle |\varphi(x) - \Pi_n \varphi(x)| \leq \sqrt{\frac{2}{\pi}} \frac{n^{-3/2}}{\sqrt{5}} \left\|\frac{d^2\varphi}{dx^2}\right\|_{L^2(0,\pi)}$

First, we prove that the inequality (ii) is true for φ replaced by v, even though v is not $\mathcal{C}^2[0,\pi]$. To see this we approximate v by $\varphi_\epsilon \in \mathcal{C}^2[0,\pi]$:

$$\varphi_\epsilon(x) = v(x) + \epsilon\, p\left(\frac{x - \left[\frac{\pi}{2} - \epsilon\right]}{\epsilon}\right)$$



where $$p(\hat{x}) = \begin{cases} \hat{x}^3 (2 - \hat{x})^3 & \text{if } \hat{x} \in [0,2], \\ 0 & \text{otherwise.} \end{cases}$$

then

$$|\varphi_\epsilon(x) - \Pi_n \varphi_\epsilon(x)| \leq \sqrt{\frac{2}{\pi}}\, \frac{n^{-1/2}}{\sqrt{3}}\, \left\|\frac{d}{dx}\varphi_\epsilon\right\|_{L^2(0,\pi)} \qquad \forall\, \epsilon < \frac{\pi}{2}.$$

But

$$\lim_{\epsilon \to 0} \varphi_\epsilon(x) = v(x)$$

$$\lim_{\epsilon \to 0} \Pi \varphi_\epsilon(x) = \lim_{\epsilon \to 0} \int_0^\pi \left[\sum_{j=1}^n \frac{\phi_j(x)\,\phi_j(y)}{\|\phi_j\|^2}\right] \varphi_\epsilon(y)\,dy$$

$$= \int_0^\pi \left[\sum_{j=1}^n \frac{\phi_j(x)\,\phi_j(y)}{\|\varphi_j\|^2}\right] \lim_{\epsilon \to 0} \varphi_\epsilon(y)\,dy$$

$$= \int_0^\pi \left[\sum_{j=1}^n \frac{\phi_j(x)\,\phi_j(y)}{\|\phi_j\|^2}\right. v(y)\,dy$$

$$= \Pi_n v$$

$$\lim_{\epsilon \to 0} \left\|\frac{d}{dx}\varphi_\epsilon\right\|_{L^2(0,\pi)} \leq \left\|\frac{d}{dx}v\right\|_{L^2(0,\pi)} + \lim_{\epsilon \to 0} \left[2\int_0^1 p^2(\hat{x})\,d\hat{x}\right]^{1/2} \epsilon^{1/2}$$

$$= \left\|\frac{dv}{dx}\right\|^2_{L^2(0,\pi)}$$

$$\Rightarrow \quad |v(x) - \Pi_n v(x)| \leq \sqrt{\frac{2}{\pi}}\, \frac{n^{-1/2}}{\sqrt{3}}\, \left\|\frac{dv}{dx}\right\|^2_{L^2(0,\pi)} \qquad \text{(iii)}$$

Now, for any $\varphi_\epsilon \in \mathcal{C}^2[0,\pi]$ such that $\varphi_\epsilon(0)=\varphi_\epsilon(\pi)$, we have that

$$|v(x) - \Pi_n v(x)| \leq |(Id - \Pi_n)(v - \varphi_\epsilon)| + |\varphi_\epsilon - \Pi_n \varphi_\epsilon|$$

$$\leq \sqrt{\frac{2}{\pi}} \cdot \frac{n^{-1/2}}{\sqrt{3}} \left\| \frac{d}{dx}(v - \varphi_\epsilon) \right\|_{L^2(0,\pi)} \quad \text{(by (iii))}$$

$$+ \sqrt{\frac{2}{\pi}} \frac{n^{-3/2}}{\sqrt{5}} \left\| \frac{d^2}{dx^2} \varphi_\epsilon \right\|_{L^2(0,\pi)} \quad \text{by (ii)}$$

This implies

$$|v(x) - \Pi_n v(x)| \leq \sqrt{\frac{2}{\pi}} \frac{n^{-1/2}}{\sqrt{3}} \left(\left\| \frac{d}{dx}(v - \varphi_\epsilon) \right\|_{L^2(0,\pi)} + n^{-1} \left\| \frac{d^2}{dx^2} \varphi_\epsilon \right\|_{L^2(0,\pi)} \right)$$

With φ_ϵ as before,

$$\left\| \frac{d}{dx}(v - \varphi_\epsilon) \right\|^2_{L^2(0,\pi)} = 2 \int_0^1 p'^2(\hat{x})\, d\hat{x} \cdot \epsilon =: c_0^2\, \epsilon$$

$$\left\| \frac{d^2}{dx^2} \varphi_\epsilon \right\|^2_{L^2(0,\pi)} = 2 \int_0^1 p''^2(\hat{x})\, d\hat{x} \cdot \epsilon^{-1} =: c_1^2\, \epsilon^{-1}$$

So,

$$|v(x) - \Pi_n v(x)| \leq \sqrt{\frac{2}{\pi}} \frac{n^{-1/2}}{\sqrt{3}} \left(c_0 \epsilon^{1/2} + c_1 n^{-1} \epsilon^{-1/2} \right) \qquad 0 < \epsilon < \pi/2$$

Taking $\epsilon = 1/n$, we get

$$|v(x) - \Pi_n v(x)| \leq \sqrt{\frac{2}{\pi}} \frac{n^{-1/2}}{\sqrt{3}} (c_0 + c_1)\, n^{-1/2}$$

$$= \sqrt{\frac{2}{3\pi}} (c_0 + c_1)\, n^{-1}.$$

3. we have that

$$\Pi_n v(x) = \sum_{j=1}^{n} \frac{\langle v, \phi_j \rangle}{\| \phi_j \|^2} \phi_j(x)$$

$$\langle v, \phi_j \rangle = \int_0^{\pi} \frac{4 \sin(x)}{\pi(5 - 4\cos(x))} \sin jx \, dx = 2^{-j}$$

$$\Rightarrow \Pi_n v(x) = \sum_{j=1}^{n} \frac{2}{\pi} 2^{-j} \sin jx$$

$$\Rightarrow \| v - \Pi_n v \|^2 = \sum_{j=n+1}^{\infty} \frac{2}{\pi} 4^{-j} = \frac{2}{4^{n+1}} \frac{1}{\pi} \sum_{j=0}^{\infty} \frac{1}{4^j} = \frac{2}{\pi} \cdot \frac{1}{4^{n+1}} \cdot \frac{1}{1 - \frac{1}{4}}$$

$$= \frac{2}{3\pi} \cdot \frac{1}{4^n}$$

$$\Rightarrow \| v - \Pi_n v \| = \sqrt{\frac{2}{3\pi}} \, 2^{-n}$$

$$= \sqrt{\frac{2}{3\pi}} \, e^{-(\ln 2) n}$$

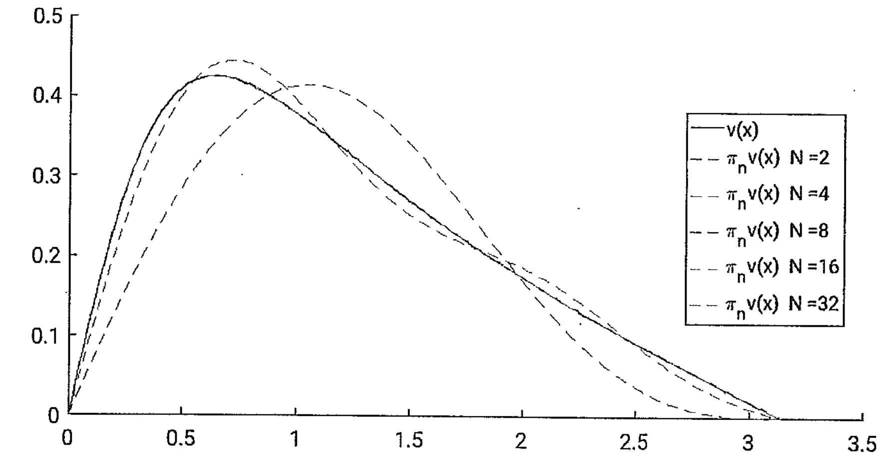
So, we should see $C_n = \sqrt{\frac{2}{3\pi}} \sim 0.46$ and $\alpha_n = \ln 2 \sim 0.6931$

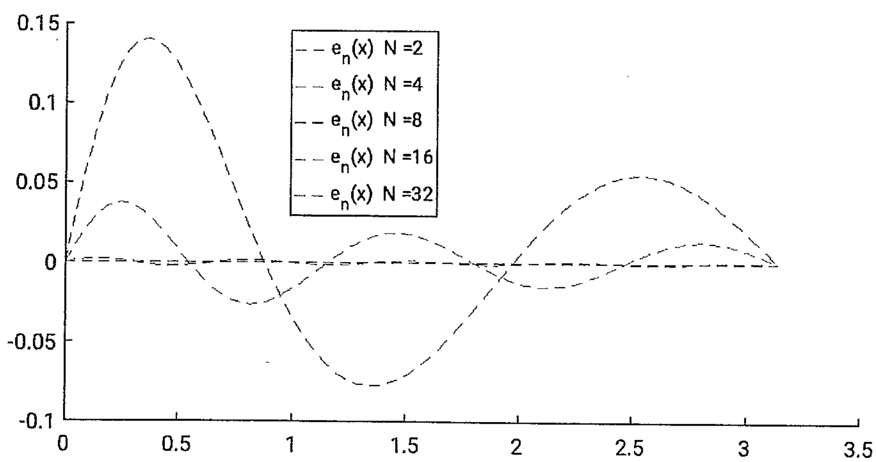
this is verified in the table below.

n	e_n	r_n	D_n
2	1.2 e-1	-	-
4	2.9 e-2	0.69	0.46
8	1.8 e-3	0.69	0.46
16	7.0 e-6	0.69	0.46
32	1.1 e-10	0.69	0.46

Table 3: History of convergence in L_2 for $v(x) = v(x) = (4\sin(x))/(\pi(5 - 4\cos(x)))$

Below we show a few $\Pi_n v(x)$ as well as the error $e_n(x) := v(x) - \Pi_n v(x)$. The convergence is so fast, we can only distinguish v from $\Pi_n v$ for $n = 2, 4, 8$.





the function v under consideration is such that

$$\left\|\frac{d^m}{dx^m} v\right\|_{L^2(0,\pi)} \leq C\, m!\, \underbrace{(2.08)}_{d}^{-m} \qquad \text{with } C = \sqrt{\frac{2}{\pi}}.$$

and so, the estimate obtained in the notes gives

$$\|\Pi_n v - v\|_{L^2(0,\pi)} \leq \frac{Ce}{\sqrt{d}}\, e^{-\frac{ne}{d}} \approx 0.93\, e^{-(0.36)\,n}$$

This estimate does not give a close approximation to $r = \ln 2$. It gives a slower exponential convergence only.

4. The history of convergence for $e_n = \|v - \Pi_n v\|_{L^\infty(0,\pi)}$ is displayed below. Notice how close is r_n from $\ln 2 \cong 0.69$. So, unlike the case of the first two problems, there is no dramatic change in the exponential rate of convergence when switching from the L^2-norm to the L^∞-norm.

n	e_n	r_n	D_n
2	1.4 e-1	-	-
4	3.7 e-2	0.66	0.52
8	2.4 e-3	0.68	0.58
16	9.6 e-6	0.69	0.61
32	1.5 e-10	0.69	0.63

Table 4: History of convergence in L_∞ for $v(x) = (4\sin(x))/(\pi(5 - 4\cos(x)))$

Since we have

$$|v(x)-\Pi_n v(x)| \le \|(Id-\Pi_n)K_{s_0}(x,\cdot)\| \quad \|(Id-\Pi_n)T^{-s_0}v\|$$

For $s_0=1$ we get

$$\|(Id-\Pi_n)K_{s_0}(x,\cdot)\| \le C_0\,(n+1)^{-3/2}$$

$$\|(Id-\Pi_n)\left(-\frac{d^2}{ds^2}v\right)\| = \left(\frac{2}{\pi}\sum_{j=n+1}^{\infty} 4^{-j}\, j^4\right)^{1/2}$$

$$= \left[\frac{27n^4+144n^3+360n^2+528n+380}{81}\right]^{1/2} 2^{-n}$$

and so

$$|v(x)-\Pi_n v(x)| \le C_0\left[\frac{n}{n+1}\right]^{3/2}\left[\frac{27+144n^{-1}+360n^{-2}+528n^{-3}+380n^{-4}}{81}\right]^{1/2} n^{-1/2}\, 2^{-n}$$

$$\le C_1\, n^{-1/2}\, 2^{-n}$$

$$= C_1\, e^{-\left[\frac{1}{2}\ln n + n\ln 2\right]}$$

$$= C_1\, e^{-n\ln 2\,\left[\underbrace{1+\frac{1}{2n}\frac{\ln n}{\ln 2}}_{\text{goes to 1 as } n \text{ goes to } \infty}\right]}$$

So, the convergence of $|v(x)-\Pi_n v(x)|$ is exponential with the same rate (as n goes to ∞) as the rate of $\|v-\Pi_n v\|$.