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<u>Homework # 6</u>

1. Consider the mapping $f \mapsto Tf =: \phi$ given by

$$-\frac{d^2}{dx^2}\phi = f \quad \text{in } (0,\pi),$$
$$\phi = 0 \quad \text{on } \{0,\pi\}.$$

we know that $T\phi_j = \kappa_j \phi_j$ if and only if

$$-\frac{d^2}{dx^2}\phi_j = \kappa_j^{-1}\phi_j \quad \text{in } (0,\pi),$$
$$\phi_j = 0 \quad \text{on } \{0,\pi\}.$$

thus, $\{\sin j\cdot\}_{j=1}^{\infty}$ is the set of all eigenvectors of $T$ and $\kappa_j = j^{-2}$. To show that $\{\sin j\cdot\}_{j=1}^{\infty}$ is complete in $L^2(0,\pi)$, we only need to show that $T$ is linear, bounded, self-adjoint and positive definite.

A simple computation gives us that

$$Tf(x) = \int_0^\pi G(x,s)\, f(s)\, ds \quad \text{where } G(x,s) = \begin{cases} \dfrac{x(\pi - s)}{\pi} & x < s, \\ \dfrac{s(\pi - x)}{\pi} & s < x. \end{cases}$$

It is immediately clear that $T$ is linear. Let us show that it is bounded. By the Cauchy-Schwarz inequality, we get that

$$|Tf(x)| \le \left(\int_0^\pi G^2(x,s)\, ds\right)^{1/2} \left(\int_0^\pi f^2(s)\, ds\right)^{1/2}$$

$$\Rightarrow \int_0^\pi |Tf(s)|^2\, ds \le \left(\int_0^\pi\int_0^\pi G^2(x,s)\, ds\, dx\right)\left(\int_0^\pi f^2(s)\, ds\right)$$

In other words

$$\|Tf\|_{L^2(0,\pi)} \leq M \|f\|_{L^2(0,\pi)}$$

where $M = \left(\int_0^\pi\int_0^\pi G^2(x,s)\,dx\,ds\right)^{1/2} = \frac{\pi^2}{\sqrt{90}}$. Hence

$$\|T\|_{\mathcal{L}(L^2(0,\pi),L^2(0,\pi))} = \sup_{f\in L^2(0,\pi)\setminus\{0\}} \frac{\|Tf\|_{L^2(0,\pi)}}{\|f\|_{L^2(0,\pi)}} \leq M$$

and so, $T$ is bounded.

Let us show that $T$ is self-adjoint. For any two elements $f_1$ and $f_2$ in $L^2(0,\pi)$, we have

$$\begin{aligned}
\langle Tf_1, f_2\rangle &= \int_0^\pi Tf_1(s)\, f_2(s)\,ds \\
&= \int_0^\pi Tf_1(s)\left(-\frac{d^2}{ds^2}Tf_2\right)(s)\,ds, \text{ by def. of } T, \\
&= \int_0^\pi -\frac{d^2}{ds^2}Tf_1(s)\cdot Tf_2(s)\,ds, \text{ by integration by parts} \\
&= \int_0^\pi f_1(s)\, Tf_2(s)\,ds \quad, \text{ by def of } T_2 \\
&= \langle f_1, Tf_2\rangle.
\end{aligned}$$

So, $T$ is self-adjoint.

Finally, let us show that $T$ is positive definite. Take any $f \in L^2(0,\pi)$. Then, proceeding as above with $f_1 = f_2 = f$

we get that

$$\langle Tf, f\rangle = \int_0^{\pi} \left(\frac{d}{ds} Tf(s)\right)^2 ds$$

$$\geq 0$$

for all $f \in L^2(0,\pi)$. If equality holds, then $\frac{d}{ds} Tf(s) = 0$ and so $Tf$ is a constant on $(0,\pi)$. But $Tf(0) = Tf(\pi) = 0$ implies that $Tf = 0$. Since $f = -\frac{d^2}{dx^2} Tf$, this implies that $f = 0$.

2. Take $T$ as the operator defined in the previous exercise but this time from $H^1_0(0,\pi)$ to $H^1_0(0,\pi)$.

Clearly, $\{\sin j\cdot\}_{j=1}^{\infty} \subseteq H^1_0(0,\pi)$.

To show that $\{\sin j\cdot\}_{j=1}^{\infty}$ is complete in $H^1_0(0,\pi)$, we must show that $T$ is linear, bounded, self-adjoint and positive-definite.

We already proved it is linear. Let us prove it is bounded. This means, we have to show there is a constant $M$ such that

$$\|T\|_{\mathcal{L}(H^1_0(0,\pi),\, H^1_0(0,\pi))} = \sup_{f \in H^1_0(0,\pi)\setminus\{0\}} \frac{\|Tf\|_{H^1_0(0,\pi)}}{\|f\|_{H^1_0(0,\pi)}} \leq M.$$

Since,

$$\|Tf\|^2_{H^1_0(0,\pi)} = \int_0^{\pi} \frac{d}{dx} Tf(x) \, \frac{d}{dx} Tf(x) \, dx$$

we have to get a suitable expression for $\frac{d}{dx} Tf$.

So

$$\frac{d}{dx} Tf(x) = \int_0^\pi \frac{\partial}{\partial x} G(x,s)\, f(s)\, ds$$

and, since $f(0)=0$ because $f \in H_0^1(0,\pi)$,

$$\frac{d}{dx} Tf(x) = \int_0^\pi \frac{\partial}{\partial x} G(x,s) \int_0^s \frac{df}{dy}(y)\, dy$$

$$= \int_0^\pi \frac{df}{dy}(y) \int_y^\pi \frac{\partial G}{\partial x}(x,s)\, ds$$

By the Cauchy-Schwarz inequality,

$$\left| \frac{d}{dx} Tf(x) \right| \leq \left( \int_0^\pi \left(\frac{df}{dy}\right)^2 (y)\, dy \right)^{1/2} \left( \int_0^\pi \left( \int_y^\pi \frac{\partial G}{\partial x}(x,s) ds \right)^2 dy \right)^{1/2}$$

$$\Rightarrow \left\| \frac{d}{dx} Tf \right\|^2_{L^2(0,\pi)} \leq \left\| \frac{df}{dy} \right\|^2_{L^2(0,\pi)} \int_0^\pi \int_0^\pi \left( \int_y^\pi \frac{\partial G}{\partial x}(x,s)\, ds \right)^2 dy\, dx$$

Since $\left| \frac{\partial G}{\partial x}(x,s) \right| \leq 1$, a simple upper bound is

$$\left\| \frac{d}{dx} Tf \right\|^2_{L^2(0,\pi)} \leq \left\| \frac{df}{dy} \right\|^2_{L^2(0,\pi)} \pi^4$$

Thus, we can take $M = \pi^2$.

Let us show that $T$ is self-adjoint. For any two $f_1, f_2 \in H_0^1(0,\pi)$, we have

$$\langle Tf_1, f_2 \rangle = \int_0^\pi \frac{d}{dx} Tf_1(x) \cdot \frac{d}{dx} f_2(x)\, dx$$

$$= \int_0^\pi \left( -\frac{d^2}{dx^2} Tf_1 \right)(x)\, f_2(x)\, dx \quad \text{since } f_2 = 0 \text{ at } 0 \text{ and } \pi,$$

$$= \int_0^\pi f_1(x) f_2(x)\, dx$$

$$= \langle f_1, T f_2 \rangle$$

It remains to show that T is positive definite. But setting $f_1 = f_2 = f$, we get that

$$\langle Tf, f \rangle = \int_0^\pi f^2(x)\, dx \geq 0$$

with equality iff $f \equiv 0$.

3. Fix $x \in (a, b)$. Let $\Pi_n G(x, \cdot)$ be the orthogonal projection of the function $y \mapsto G(x, y)$ into $W_n := \operatorname{span}\{\sin j \cdot\}_{j=1}^n$. Then

$$\Pi_n G(x, \cdot)(y) = \sum_{j=1}^n \frac{\langle G(x, \cdot), \phi_j \rangle}{\pi/2} \phi_j(y)$$

But

$$\langle G(x, \cdot)\ \phi_j \rangle = \int_a^b G(x, s)\, \phi_j(s)\, ds = (T\phi_j)(x) = \kappa_j \phi_j(x).$$

then

$$\Pi_n G(x, \cdot)(y) = \sum_{j=1}^n \kappa_j \frac{\phi_j(x)\, \phi_j(y)}{\|\phi_j\|^2}$$

By Bessel's inequality

$$\|\Pi_n G(x, \cdot)\|^2_{L^2(a,b)} = \sum_{j=1}^n \kappa_j^2 \frac{\phi_j^2(x)}{\|\phi_j\|^2} \leq \int_a^b G^2(x, y)\, dy$$

then

$$\int_a^b \| \Pi_n G(x,\cdot) \|^2_{L(a,b)} \, dx = \sum_{j=1}^{n} k_j^2 \leq \int_a^b \int_a^b G^2(x,s) \, dx \, ds$$

This is true for any n. Hence

$$\sum_{j=1}^{\infty} k_j^2 \leq \int_a^b \int_a^b G^2(x,s) \, dx ds < \infty$$

implies that $k_j$ must converge to zero as j goes to infinity.

4. By definition, $P_n u$ is the element of $W_n := \text{span}\{\sin j\cdot\}_{j=1}^{n}$ such that

$$\int_0^\pi \frac{d}{dx} P_n u \cdot \frac{d}{dx} \sin jx \, dx = \int_0^\pi f(x) \sin jx \, dx \qquad j = 1, \dots, n.$$

Since $f = -\frac{d^2}{dx^2} u$ and since $u = 0$ at $x=0$, $x=\pi$, we have that

(*) $$\int_0^\pi \frac{d}{dx} P_n u \cdot \frac{d}{dx} \sin jx = \int_0^\pi \frac{du}{dx} \cdot \frac{d}{dx} \sin jx \, dx \qquad j = 1, \dots, n$$

or, equivalently,

$$\langle P_n u, \sin j\cdot \rangle = \langle u, \sin j\cdot \rangle \qquad j = 1, \dots, n$$

$$\Rightarrow \quad \langle P_n u - u, w \rangle = 0 \qquad \forall \; w \in W_n$$

In other words, $P_n$ is the orthogonal projection of $u \in H_0^1(0,\pi)$ into $W_n$ (with inner product $\langle f_1, f_2 \rangle = \int_0^\pi \frac{df_1}{dx} \cdot \frac{df_2}{dx}$.)

Integration by parts in (*) gives

$$\int_0^\pi P_n u(x)\left(-\frac{d^2}{dx^2}\sin jx\right)dx = \int_0^\pi u(x)\left(-\frac{d}{dx^2}\sin jx\right)dx \qquad j=1,\dots,n$$

or, equivalently,

$$\int_0^\pi P_n u(x)\sin jx\,dx = \int_0^\pi u(x)\sin jx\,dx \qquad j=1,\dots,n$$

which means that $P_n u = \Pi_n u$, as wanted.