MATH 8441 . Homework #8 . A solution

We consider the problem of approximating a function u on the domain $(0,1)$ by the L^2-projection into

$$W_h = \{ w \in \mathcal{C}^0(0,1) : w|_{I_i} \in P_k(I_i),\ i=1,2,\dots,N \}$$

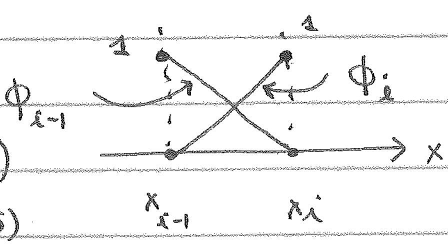
1. For $k=1$, W_h is the space of continuous functions which are linear on each of the intervals I_i. Since there are N of those intervals, the dimension of W_h is $N+1$. then

$$W_h = \text{span}\{\phi_j\}_{j=0}^{N},$$

where we take $\phi_j \in W_h$ such that $\phi_j(x_i) = \delta_{ji}$ for $0 \le i,j \le N$. this implies that, if

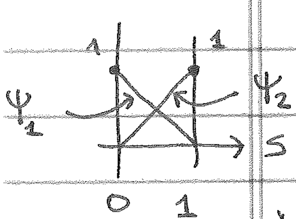
$$s = \frac{x - x_{i-1}}{x_i - x_{i-1}} \qquad \forall\, x \in I_i = (x_{i-1}, x_i)$$

then, for $x \in I_i$:



$$\phi_i(x) = s =: \psi_2(s)$$

$$\phi_{i-1}(x) = 1 - s =: \psi_1(s)$$



Now that we have found a basis for W_h, we can find the matrix equation defining the degrees of freedom of $\Pi_h u$.

Since $\Pi_h u \in W_h$, we can write

$$(*) \qquad \Pi_h u(x) = \sum_{j=0}^{N} c_j \phi_j(x)$$

Here, $\{c_j\}_{j=0}^{N}$ is the set of degrees of freedom of $\Pi_h u$. To find what matrix equation they satisfy, we begin by rewriting the definition of $\Pi_h u$,

$$\int_0^1 \Pi_h u \cdot w = \int_0^1 u \cdot w \qquad \forall\, w \in W_h$$

as

$$\int_0^1 \Pi_h u \cdot \phi_i = \int_0^1 u \, \phi_i \qquad i = 0, \dots, N.$$

Then, we insert the form of $\Pi_h u$ given by $(*)$ to get

$$\sum_{j=0}^{N} \left[\int_0^1 \phi_i \phi_j \right] c_j = \int_0^1 u \phi_i \qquad i = 0, \dots, N.$$

On any given element I_ℓ, we have to compute

$$\int_{I_\ell} \phi_i \phi_j \qquad \text{and} \qquad \int_{I_\ell} u \, \phi_i \, .$$

The only values of i and j for which these integrals are not zero are $\underline{\ell-1 \text{ and}}$ ℓ. So we have to compute

$$\underbrace{\begin{bmatrix} \int_{I_\ell} \phi_{\ell-1} \phi_{\ell-1} & \int_{I_\ell} \phi_{\ell-1} \phi_\ell \\ \int_{I_\ell} \phi_\ell \phi_{\ell-1} & \int_{I_\ell} \phi_\ell \phi_\ell \end{bmatrix}}_{A_{I_\ell}} \quad \text{and} \quad \underbrace{\begin{bmatrix} \int_{I_\ell} \phi_{\ell-1} u \\ \int_{I_\ell} \phi_\ell u \end{bmatrix}}_{b_\ell}$$

But

$$\int_{I_\ell} \phi_{\ell-1}(x)\,\phi_{\ell-1}(x)\,dx = \int_0^1 \psi_1(s)\,\psi_1(s)\cdot ds \cdot (x_\ell - x_{\ell-1})$$

$$\int_{I_\ell} \phi_{\ell-1}(x)\,\phi_{\ell}(x)\,dx = \int_0^1 \psi_1(s)\,\psi_2(s)\cdot ds \cdot (x_\ell - x_{\ell-1})$$

$$\int_{I_\ell} \phi_{\ell}(x)\,\phi_{\ell}(x)\,dx = \int_0^1 \psi_2(s)\,\psi_2(s)\cdot ds \cdot (x_\ell - x_{\ell-1})$$

and, for $u(x) = \sin \pi x$

$$\int_{I_\ell} \phi_{\ell-1}(x)\,\sin(\pi x)\,dx = \int_{x_{\ell-1}}^{x_\ell} \frac{x_\ell - x}{x_\ell - x_{\ell-1}} \cdot \sin(\pi x)\,dx$$

$$\int_{I_\ell} \phi_{\ell}(x)\,\sin(\pi x)\cdot dx = \int_{x_{\ell-1}}^{x_\ell} \frac{x - x_{\ell-1}}{x_\ell - x_{\ell-1}} \cdot \sin(\pi x)\,dx$$

After a small calculation we get

$$A_\ell = (x_\ell - x_{\ell-1}) \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

$$b_\ell = \begin{bmatrix} \frac{1}{\pi}\cos \pi x_{\ell-1} - \frac{1}{\pi^2}\left(\frac{\sin \pi x_\ell - \sin \pi x_{\ell-1}}{x_\ell - x_{\ell-1}}\right) \\ -\frac{1}{\pi}\cos \pi x_\ell + \frac{1}{\pi^2}\left(\frac{\sin \pi x_\ell - \sin \pi x_{\ell-1}}{x_\ell - x_{\ell-1}}\right) \end{bmatrix}$$

For $x_\ell - x_{\ell-1} = h$, we get

$$A_\ell = \frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$b_\ell = \begin{bmatrix} \dfrac{\cos \pi h(\ell-1)}{\pi} - \dfrac{1}{h\pi^2}\left(\sin \pi h \ell - \sin \pi h(\ell-1)\right) \\ -\dfrac{\cos \pi h \ell}{\pi} + \dfrac{1}{h\pi^2}\left(\sin \pi h \ell - \sin \pi h(\ell-1)\right) \end{bmatrix} = \begin{bmatrix} b_{\ell 1} \\ b_{\ell 2} \end{bmatrix}$$

Since

$$b_{\ell 2} + b_{(\ell+1)1} = \frac{1}{h\pi^2}\left(-\sin \pi h(\ell+1) + 2\sin \pi h \ell - \sin \pi h(\ell-1)\right)$$

$$= \frac{1}{h\pi^2}\left(2\sin \pi h \ell\right)\left(1 - \cos \pi h\right)$$

$$= \frac{1}{h\pi^2}\left(2\sin \pi h \ell\right)\left(2\sin^2 \frac{\pi h}{2}\right)$$

$$= \left(\frac{\sin \frac{\pi h}{2}}{\frac{\pi h}{2}}\right)^2 \sin \pi h \ell$$

$$b_{11} = \frac{1}{\pi} - \frac{1}{h\pi^2}\sin \pi h$$

$$= \frac{1}{\pi}\left(1 - \frac{\sin \pi h}{\pi h}\right)$$

$$b_{N2} = \frac{1}{\pi} - \frac{1}{h\pi^2}\sin \pi h(N-1)$$

$$= \frac{1}{\pi} - \frac{1}{h\pi^2}\sin\left(\pi[1-h]\right)$$

$$= \frac{1}{\pi}\left(1 - \frac{1}{h\pi}\sin h\pi\right)$$

we get

$$\frac{h}{6}\begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 4 & 1 & & & & & \\ & 1 & 4 & 1 & & & & \\ & & 1 & 4 & \ddots & & & \\ & & & \ddots & \ddots & & & \\ & & & & 4 & 1 & & \\ & & & & 1 & 4 & 1 & \\ & & & & & 1 & 4 & 1 \\ & & & & & & 1 & 2 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{N-3} \\ c_{N-2} \\ c_{N-1} \\ c_N \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \sin \pi h \\ \beta \sin 2\pi h \\ \beta \sin 3\pi h \\ \vdots \\ \beta \sin (N-3)\pi h \\ \beta \sin (N-2)\pi h \\ \beta \sin (N-1)\pi h \\ \alpha \end{bmatrix}$$

where $\alpha = \frac{1}{\pi}\left(1 - \frac{\sin \pi h}{\pi h}\right)$ and $\beta = \left(\frac{\sin \frac{h\pi}{2}}{\frac{h\pi}{2}}\right)^2$.

For $\Pi_h u$ we have shown that

$$\|u - \Pi_h u\|_{L^2(0,1)} \leq C \frac{1}{N^2} \left\|\frac{d^2 u}{dx^2}\right\|_{L^2(0,1)},$$

and so, we expect to see α_n tending to 2 in the table of the history of convergence of the method.

n	$e_n = \|u - \Pi_h u\|_{L^2(0,1)}$	α_n	C_n
5	$1.08\ 10^{-2}$	–	–
10	$2.63\ 10^{-3}$	2.03	0.282
20	$6.52\ 10^{-4}$	2.01	0.269
40	$1.63\ 10^{-4}$	2.00	0.261
80	$4.06\ 10^{-5}$	2.00	0.260

We see that we do get what was expected theoretically.

2. For $k>1$, the functions in W_h restricted to the interval I_i lie in the space $P_k(I_i)$. We take the basis functions of the previous exercise and add $k-1$ functions. The additional functions are zero at the boundary of the interval I_i so that their extension by zero to the whole domain $(0,1)$ is a continuous function.

We are going to express the element of the local basis in terms of Legendre polynomials. To do that, we use the mapping $x \in I_i \longmapsto s = \left(x - \frac{x_i + x_{i-1}}{2}\right) / \left(\frac{x_i - x_{i-1}}{2}\right) \in (-1,1)$. Then, the local basis functions for $k=1$ are:

$$\phi_{i-1}(x) = \tfrac{1}{2}(P_0(s) - P_1(s)) =: \psi_1(s)$$
$$\phi_i(x) = \tfrac{1}{2}(P_0(s) + P_1(s)) =: \psi_{k+1}(s)$$

The local basis functions we have to add are

$$b_{2i}(x) = P_2(s) - P_0(s) =: \psi_2(s)$$
$$b_{3i}(x) = P_3(s) - P_1(s) =: \psi_3(s)$$
$$b_{4i}(x) = P_4(s) - P_0(s) =: \psi_4(s)$$
$$b_{5i}(x) = P_5(s) - P_1(s) =: \psi_5(s)$$
$$\dots$$
$$b_{ki}(x) = P_k(s) - P_l(s) =: \psi_k(s), \quad k = 2m+l$$

Then, the local matrices are

$$M_i := \begin{bmatrix} \langle \phi_{i-1}, \phi_{i-1} \rangle & \langle \phi_{i-1}, b_{2i} \rangle & \langle \phi_{i-1}, b_{3i} \rangle & \cdots & \langle \phi_{i-1}, b_{ki} \rangle & \langle \phi_{i-1}, \phi_i \rangle \\ \langle b_{2i}, \phi_{i-1} \rangle & \langle b_{2i}, b_{2i} \rangle & \langle b_{2i}, b_{3i} \rangle & \cdots & \langle b_{2i}, b_{ki} \rangle & \langle b_{2i}, \phi_i \rangle \\ \langle b_{3i}, \phi_{i-1} \rangle & \langle b_{3i}, b_{2i} \rangle & \langle b_{3i}, b_{3i} \rangle & \cdots & \langle b_{3i}, b_{ki} \rangle & \langle b_{3i}, \phi_i \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle b_{ki}, \phi_{i-1} \rangle & \langle b_{ki}, b_{2i} \rangle & \langle b_{ki}, b_{3i} \rangle & \cdots & \langle b_{ki}, b_{ki} \rangle & \langle b_{ki}, \phi_i \rangle \\ \langle \phi_i, \phi_{i-1} \rangle & \langle \phi_i, b_{2i} \rangle & \langle \phi_i, b_{3i} \rangle & \cdots & \langle \phi_i, b_{ki} \rangle & \langle \phi_i, \phi_i \rangle \end{bmatrix}$$

and

$$b_i := \begin{bmatrix} \langle u, \phi_{i-1} \rangle \\ \langle u, b_{2i} \rangle \\ \langle u, b_{3i} \rangle \\ \vdots \\ \langle u, b_{ki} \rangle \\ \langle u, \phi_i \rangle \end{bmatrix}$$

If we denote $\int_{-1}^{1} a(s)\, b(s)\, ds$ by (a, b), we have

$$M_i = \frac{(x_i - x_{i-1})}{2} \begin{bmatrix} (\psi_1, \psi_1) & (\psi_1, \psi_2) & (\psi_1, \psi_3) & \cdots & (\psi_1, \psi_k) & (\psi_1, \psi_{k+1}) \\ (\psi_2, \psi_1) & (\psi_2, \psi_2) & (\psi_2, \psi_3) & \cdots & (\psi_2, \psi_k) & (\psi_2, \psi_{k+1}) \\ (\psi_3, \psi_1) & (\psi_3, \psi_2) & (\psi_3, \psi_3) & \cdots & (\psi_3, \psi_k) & (\psi_3, \psi_{k+1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (\psi_k, \psi_1) & (\psi_k, \psi_2) & (\psi_k, \psi_3) & \cdots & (\psi_k, \psi_k) & (\psi_k, \psi_{k+1}) \\ (\psi_{k+1}, \psi_1) & (\psi_{k+1}, \psi_2) & (\psi_{k+1}, \psi_3) & \cdots & (\psi_{k+1}, \psi_k) & (\psi_{k+1}, \psi_{k+1}) \end{bmatrix}$$

and

$$b_i = \begin{bmatrix} \int_{x_{i-1}}^{x_i} \frac{1}{2}(1-s) \sin \pi x \, dx \\ \int_{x_{i-1}}^{x_i} (P_2(s) - 1) \sin \pi x \, dx \\ \int_{x_{i-1}}^{x_i} (P_3(s) - s) \sin \pi x \, dx \\ \cdots \\ \int_{x_{i-1}}^{x_i} \frac{1}{2}(1+s) \sin \pi x \, dx \end{bmatrix} \qquad s = \frac{x - \frac{x_i + x_{i-1}}{2}}{\left(\frac{x_i - x_{i-1}}{2}\right)}.$$

For $h = (x_i - x_{i-1})$ and $k=3$, we get

$$M_i = \frac{h}{2}\begin{bmatrix} \frac{2}{3} & -1 & \frac{1}{3} & \frac{1}{3} \\ -1 & \frac{12}{5} & 0 & -1 \\ \frac{1}{3} & 0 & \frac{20}{21} & -\frac{1}{3} \\ \frac{1}{3} & -1 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$b_i = \begin{bmatrix} \frac{1}{\pi}\cos\pi x_{i-1} - \frac{1}{h\pi^2}\left(\sin\pi x_i - \sin\pi x_{i-1}\right) \\ \frac{6}{h\pi^2}\left[\sin\pi x_i + \sin\pi x_{i-1}\right] + \frac{12}{h^2\pi^3}\left[\cos\pi x_i - \cos\pi x_{i-1}\right] \\ \left[\frac{10}{h\pi^2} - \frac{120}{h^3\pi^4}\right]\left[\sin\pi x_i - \sin\pi x_{i-1}\right] + \frac{60}{h^2\pi^3}\left[\cos\pi x_i + \cos\pi x_{i-1}\right] \\ -\frac{1}{\pi}\cos\pi x_i + \frac{1}{h\pi^2}\left(\sin\pi x_i - \sin\pi x_{i-1}\right) \end{bmatrix} = \begin{bmatrix} b_{i1} \\ b_{i2} \\ b_{i3} \\ b_{i4} \end{bmatrix}$$

and the matrix equation is, for $h = \frac{1}{5}$,

$$\frac{1}{10}\begin{bmatrix} \frac{2}{3} & -1 & \frac{1}{3} & \frac{1}{3} & & & & & & & \\ -1 & \frac{12}{5} & 0 & -1 & & & & & & & \\ \frac{1}{3} & 0 & \frac{20}{21} & -\frac{1}{3} & & & & & & & \\ \frac{1}{3} & -1 & -\frac{1}{3} & \frac{4}{3} & -1 & \frac{1}{3} & \frac{1}{3} & & & & \\ & & & -1 & \frac{12}{5} & 0 & -1 & & & & \\ & & & \frac{1}{3} & 0 & \frac{20}{21} & -\frac{1}{3} & & & & \\ & & & \frac{1}{3} & -1 & -\frac{1}{3} & \frac{4}{3} & & & & \\ & & & & & & & \ddots & & & \\ & & & & & & & \frac{4}{3} & -1 & \frac{1}{3} & \frac{1}{3} \\ & & & & & & & -1 & \frac{12}{5} & 0 & -1 \\ & & & & & & & \frac{1}{3} & 0 & \frac{20}{21} & -\frac{1}{3} \\ & & & & & & & \frac{1}{3} & -1 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} c_0 \\ c_{12} \\ c_{13} \\ c_1 \\ c_{22} \\ c_{23} \\ c_2 \\ \vdots \\ c_4 \\ c_{42} \\ c_{43} \\ c_5 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \\ b_{14} + b_{21} \\ b_{22} \\ b_{23} \\ b_{24} + b_{31} \\ \vdots \\ b_{44} + b_{51} \\ b_{52} \\ b_{53} \\ b_{54} \end{bmatrix}$$

M_1 M_2 M_3 M_4 M_5

b_1 b_2 b_3 b_4 b_5

Here we are assuming the notation:

$$\Pi_h u(x) = \sum_{j=0}^{N} c_j \phi_j(x) + \sum_{i=1}^{N} \sum_{l=2}^{k} c_{il}\, b_{il}(x)$$

sum over the "nodes" x_j

sum over the intervals

sum over the "bubble" functions.

3. We write W_h as the sum of W_h^∂ and W_h^b where

$$W_h^\partial := \text{span}\,\{\phi_j\}_{j=0}^{N}$$

$$W_h^b := \text{span}\,\{b_{il} : l=2,\ldots,k\}_{i=1}^{N}$$

then $$(\Pi_h u)^\partial := \prod_{j=0}^{N} c_j\,\phi_j(x)$$

$$(\Pi_h u)^b := \sum_{i=1}^{N}\sum_{l=2}^{k} c_{il}\, b_{ij}(x)$$

the definition of $\Pi_h u = (\Pi_h u)^\partial + (\Pi_h u)^b$ is

$$\langle(\Pi_h u)^b, \omega^b\rangle + \langle(\Pi_h u)^\partial, \omega^b\rangle = \langle u, \omega^b\rangle \quad \forall\, \omega^b \in W_h^b$$
$$\langle(\Pi_h u)^b, \omega^\partial\rangle + \langle(\Pi_h u)^\partial, \omega^\partial\rangle = \langle u, \omega^\partial\rangle \quad \forall\, \omega^\partial \in W_h^\partial$$

Since $\langle(\Pi_h u)^b, \omega^\partial\rangle = \langle(\Pi_h u)^b, \Pi_h^b\,\omega^\partial\rangle$, where Π_h^b denotes the orthogonal projection into the space of bubbles W_h^b, we have, from the first equation with $\omega^b := \Pi_h^b\,\omega^b$, that

$$\langle(\Pi_h u)^b, \omega^\partial\rangle = \langle u, \Pi_h^b\,\omega^\partial\rangle - \langle(\Pi u)^\partial, \Pi_h^b\,\omega^\partial\rangle$$

Inserting this expression into the second equation, we get

$$\langle(\Pi u)^\partial, \omega^\partial - \Pi_h^b\,\omega^\partial\rangle = \langle u, \omega^\partial - \Pi_h^b\,\omega^\partial\rangle \qquad \forall\, \omega \in W_h^\partial$$

the local matrices for the above weak formulation are

$$\overbrace{\begin{bmatrix} \langle \phi_{i-1} - \Pi_h^b \phi_{i-1}, \phi_{i-1} \rangle & \langle \phi_{i-1} - \Pi_h^b \phi_{i-1}, \phi_i \rangle \\ \langle \phi_i - \Pi_h^b \phi_i, \phi_{i-1} \rangle & \langle \phi_i - \Pi_h^b \phi_i, \phi_i \rangle \end{bmatrix}}^{M_i}, \overbrace{\begin{bmatrix} \langle \phi_{i-1} - \Pi_h^b \phi_{i-1}, u \rangle \\ \langle \phi_i - \Pi_h^b \phi_i, u \rangle \end{bmatrix}}^{b_i}.$$

Let us find these matrices for $h = (x_i - x_{i-1})$ and $k = 3$.
Since b_{2i} and b_{3i} are orthogonal, we can write

$$\Pi_h^b \phi_{i-1} = \phi_{i-1} - \frac{\langle \phi_{i-1}, b_{2i} \rangle}{\langle b_{2i}, b_{2i} \rangle} b_{2i} - \frac{\langle \phi_{i-1}, b_{3i} \rangle}{\langle b_{3i}, b_{3i} \rangle} b_{3i}$$

$$\Pi_h^b \phi_i = \phi_i - \frac{\langle \phi_i, b_{2i} \rangle}{\langle b_{2i}, b_{2i} \rangle} b_{2i} - \frac{\langle \phi_i, b_{3i} \rangle}{\langle b_{3i}, b_{3i} \rangle} b_{3i}$$

then we have that

$$M_i = \begin{bmatrix} \frac{\langle \phi_{i-1}, b_{2i} \rangle^2}{\langle b_{2i}, b_{2i} \rangle} + \frac{\langle \phi_{i-1}, b_{3i} \rangle^2}{\langle b_{3i}, b_{3i} \rangle} & \frac{\langle \phi_{i-1}, b_{2i} \rangle \langle \phi_i, b_{2i} \rangle}{\langle b_{2i}, b_{2i} \rangle} + \frac{\langle \phi_{i-1}, b_{3i} \rangle \langle \phi_i, b_{3i} \rangle}{\langle b_{3i}, b_{3i} \rangle} \\ \frac{\langle \phi_i, b_{2i} \rangle \langle \phi_{i-1}, b_{2i} \rangle}{\langle b_{2i}, b_{2i} \rangle} + \frac{\langle \phi_i, b_{3i} \rangle \langle \phi_{i-1}, b_{3i} \rangle}{\langle b_{3i}, b_{3i} \rangle} & \frac{\langle \phi_i, b_{2i} \rangle^2}{\langle b_{2i}, b_{2i} \rangle} + \frac{\langle \phi_i, b_{3i} \rangle^2}{\langle b_{3i}, b_{3i} \rangle} \end{bmatrix}$$

$$b_i = \begin{bmatrix} \frac{\langle \phi_{i-1}, b_{2i} \rangle}{\langle b_{2i}, b_{2i} \rangle} & \frac{\langle \phi_{i-1}, b_{3i} \rangle}{\langle b_{3i}, b_{3i} \rangle} \\ \frac{\langle \phi_i, b_{2i} \rangle}{\langle b_{2i}, b_{2i} \rangle} & \frac{\langle \phi_i, b_{3i} \rangle}{\langle b_{3i}, b_{3i} \rangle} \end{bmatrix} \underbrace{\begin{bmatrix} \langle b_{2i}, u \rangle \\ \langle b_{3i}, u \rangle \end{bmatrix}}_{\text{already computed in ex. 2}}$$

then, we have that the local matrices are

$$M_i = \frac{h}{60}\begin{bmatrix} 16 & 9 \\ 9 & 16 \end{bmatrix}, \quad b_i = \begin{bmatrix} -5/12 & 7/20 \\ -5/12 & -7/20 \end{bmatrix}\begin{bmatrix} \langle b_{2i}, w\rangle \\ \langle b_{3i}, w\rangle \end{bmatrix}$$

4. The history of convergence is below. Since $k=3$, we expect convergence of order N^{-4}, which is what we see

n	e_n	α_n	C_n
2	$8.30\ 10^{-4}$	–	–
4	$5.50\ 10^{-5}$	3.92	$1.25\ 10^{-2}$
8	$3.40\ 10^{-6}$	4.02	$1.44\ 10^{-2}$
16	$2.10\ 10^{-7}$	4.02	$1.44\ 10^{-2}$
32	$1.30\ 10^{-8}$	4.02	$1.44\ 10^{-2}$