Happy HalloWEIERSTRASS

Did you know that Weierestrass
was born on Halloween?
Neither did we...

Dmitriy Bilyk

will be speaking on

Lacunary Fourier series:
from Weierstrass
to our days

Monday, Oct 31

at **12:15pm** in **Vin 313**
followed by **Mesa Pizza**
in the first floor lounge

Brought to you by the
UMN AMS Student Chapter and

Student Unions & Activities

Karl Theodor Wilhelm Weierstrass

31 October 1815 – 19 February 1897

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- often cited as the *father of modern analysis*

Doctoral students of Karl Weierstrass include

- Georg Cantor
- Georg Frobenius
- Sofia Kovalevskaya
- Carl Runge
- Hans von Mangoldt
- Hermann Schwarz
- Magnus Gustaf (Gösta) Mittag-Leffler*

Weierstrass's doctoral advisor was Christoph Gudermann, a student of Carl Gauss.

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- Weierstrass Institute for Applied Analysis and Stochastics (Berlin)

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- trajectories of stochastic processes

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  - proved that trajectories are a.s. nowhere differentiable (with Paley and Zygmund).

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- smoothness of f "$\Longleftrightarrow$" decay of $\widehat{f}_n$

What does “lacunary” mean?

- **lacuna** (*noun, plural:* lacunae, lacunas)
  - [luh-**kyoo**-nuh]
  - a gap or a missing part, as in a manuscript, series, or logical argument.
  - from *Latin* **lacuna**: ditch, pit, hole, gap, akin to **lacus**: lake.
  - cf. *English* lagoon, lake.

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  - cf. *English* lagoon, lake.
- **lacunary** (*adjective*)
  - [**lak**-yoo-ner-ee, luh-**kyoo**-nuh-ree]
  - having lacunae.

Lacunary sequences

- A sequence $(n_k) \subset \mathbb{N}$ is called (Hadamard) **lacunary** if for some $\lambda > 1$ and for all $k \in \mathbb{N}$:

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- other lacunarities: e.g., (n^2) or $(n!)$
- **Lacunary Fourier (trigonometric) series** are series of the form

$$\sum_{k=1}^{\infty} c_k e^{2\pi i n_k x} \quad \text{or} \quad \sum_{k=1}^{\infty} a_k \sin(2\pi n_k x + \phi_k),$$

  where (n_k) is a lacunary sequence.

Riemann's remark

Quote from Weierstrass:
Erst Riemann hat, wie ich von einigen seiner Zuhörer erfahren habe, mit Bestimmtheit ausgesprochen (i.J. 1861, oder vielleicht schon früher), dass jene Annahme unzulässig sei, und z.B. bei der durch die unendliche Reihe

$$\sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}$$

dargestellten Function sich nicht bewahrheite. Leider ist der Beweis hierfür von Riemann nicht veröffentlicht worden, und scheint sich auch nicht in seinen Papieren oder mündlich Uberlieferung erhalten zu haben. Dieses ist um so mehr zu bedauern, als ich nicht einmal mit Sicherheit habe erfahren können, wie Riemann seinen Zuhörern gegenüber sich ausgedrückt hat.

Weierstrass function

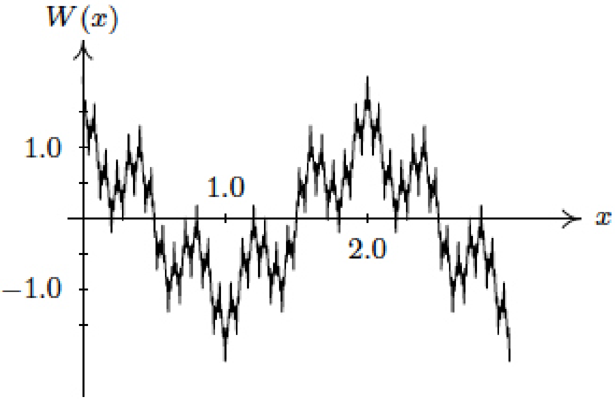
Theorem

Let $0 < a < 1$, $b > 1$. *The function*

$$\sum_{n=1}^{\infty} a^n \cos(b^n x)$$

is continuous and nowhere differentiable

- *if* $ab > 1 + \frac{3\pi}{2}$, b *an odd integer (Weierstrass, 1872)*
- *if* $ab > 1$ *(du Bois-Reymond, 1875)*
- *if* $ab \geq 1$ *(Hardy, 1916)*



Weierstrass function with $a = 0.5$ and $b = 5$

Uniformity of behavior

Assume that f has lacunary Fourier series $\sum_k a_k \cos(n_k x + \phi_k)$ with $\frac{n_{k+1}}{n_k} > \lambda > 1$, $\sum |a_k| < 1$.

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  (c) f satisfies (a) uniformly for all t_0.

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  - $\frac{2p+1}{2q}$, p, $q \in \mathbb{Z}$.
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  with derivative $-\frac{1}{2}$.

Hadamard: analytic continuation

Theorem (Hadamard, 1892)

If (n_k) *is lacunary, i.e.* $\frac{n_{k+1}}{n_k} \geq q > 1$, *and* $\limsup_{k\to\infty} |a_k|^{1/n_k} = 1$, *then the Taylor series*

$$\sum_{k=1}^{\infty} a_k z^{n_k}$$

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Analogs for lacunary Fourier series

- Kolmogorov (1924):
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Sidon's theorems

Assume that f has lacunary Fourier series $\sum_k a_k \sin(n_k x + \phi_k)$ with $\frac{n_{k+1}}{n_k} > \lambda > 1$, $\sum |a_k| < 1$.

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- for all $p \in [1, \infty)$,

$$c_p \|f\|_2 \leq \|f\|_p \leq C_p \|f\|_2.$$

Probabilistic analogs

Let $\{r_n\}$ be random signs, i.e. independent random variables on a probability space Ω with $\mathbb{P}(r_n = +1) = \mathbb{P}(r_n = -1) = \frac{1}{2}$.

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- Obvious:

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- Khinchine inequality (1923):

  For $0 < p < \infty$,

$$c_p \Big(\sum |a_n|^2 \Big)^{1/2} \leq \left(\mathbb{E} \Big| \sum a_n r_n \Big|^p \right)^{1/p} \leq C_p \Big(\sum |a_n|^2 \Big)^{1/2}.$$