Adaptive Wavelet Methods for Linear–Quadratic Elliptic Control Problems

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Dirichlet Problem with Distributed Control

given y_* and f
$\omega > 0$

$$\begin{aligned} \text{minimize} \quad J(y,u) &= \frac{1}{2}\|y - y_*\|_Y^2 \quad + \frac{\omega}{2}\|u\|_Q^2 \\ \text{subject to} \ -\Delta y &= f + u \qquad \text{in } \Omega \subset \mathbb{R}^d \\ y &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

$A : H_0^1(\Omega) \to H^{-1}(\Omega)$
$\langle Av, w \rangle := \int_\Omega \nabla v \cdot \nabla w \, dx$

set $Y := H_0^1(\Omega)$ $\quad Q := Y'$
y state $\quad u$ control

$\rightsquigarrow$ weak formulation

given y_* and f
$\omega > 0$

$$\begin{aligned} \text{minimize} \quad J(y,u) &= \frac{1}{2}\|y - y_*\|_Y^2 \quad + \frac{\omega}{2}\|u\|_Q^2 \\ \text{subject to} \ Ay &= f + u \end{aligned}$$

Classical Approach for PDE Constraints

$A : Y \to Y'$ isomorphism
$\rightsquigarrow$ discretization $\rightsquigarrow$ finite dim. problem $\rightsquigarrow$ fast solvers

obstructions: large ill–conditioned system

New Paradigm

(I) mapping property for $A : Y \to Y'$

(II) transformation into equivalent ∞–dimensional well–posed ℓ_2 problem

(III) convergent iteration for the ∞–dimensional ℓ_2 problem

(IV) adaptive application of operator A

(V) convergence/complexity analysis

$\rightsquigarrow$ set up control problem in wavelet coordinates

(Biorthogonal) Wavelets

$\mathcal{H}$ Hilbert space with $\|\cdot\|_{\mathcal{H}}$ — $\mathcal{H}'$ dual of $\mathcal{H}$ with $\langle\cdot,\cdot\rangle$

$\Psi := \{\psi_\lambda : \lambda \in \mathcal{I}\} \subset \mathcal{H}$ Wavelets — $\mathcal{I}$ (infinite) index set

(P1) Riesz basis property

$v \in \mathcal{H}$: $\quad v \; = \; \mathbf{v}^T\Psi \; := \; \sum_{\lambda\in\mathcal{I}} v_\lambda \, \psi_\lambda$ such that $\|v\|_{\mathcal{H}} \;\sim\; \|\mathbf{v}\|_{\ell_2(\mathcal{I})}$

(P2) locality $\quad \mathrm{diam}\,(\mathrm{supp}\,\psi_\lambda) \;\sim\; 2^{-|\lambda|}$ — $|\lambda|$ resolution

ψ_λ centered around $2^{-|\lambda|}k$

(P1) + duality $\quad\Longrightarrow\quad$ for every $\tilde{v} \in \mathcal{H}'$:

$$\tilde{v} = \tilde{\mathbf{v}}^T\tilde{\Psi} := \langle\tilde{v},\Psi\rangle\tilde{\Psi} := \sum_{\lambda\in\mathcal{I}} \langle\tilde{v},\psi_\lambda\rangle\tilde{\psi}_\lambda \quad \text{such that} \quad \|\tilde{v}\|_{\mathcal{H}'} \;\sim\; \|\tilde{\mathbf{v}}\|_{\ell_2(\mathcal{I})}$$

(P3) biorthogonality $\quad \langle\psi_\lambda,\tilde{\psi}_\mu\rangle = \delta_{\lambda,\mu} \qquad \lambda,\mu\in\mathcal{I}$

Representer for Control Problem in Wavelet Coordinates

minimize $\quad \mathbf{J}(\mathbf{y},\mathbf{u}) \;=\; \frac{1}{2}\|\mathbf{y}-\mathbf{y}_*\|^2 + \frac{\omega}{2}\|\mathbf{u}\|^2 \qquad \|\cdot\| := \|\cdot\|_{\ell_2}$

subject to $\quad \mathbf{A}\mathbf{y} = \mathbf{f} + \mathbf{u} \qquad \mathbf{A} : \ell_2 \to \ell_2$ automorphism

Necessary and Sufficient Conditions

$$\mathbf{Lagr}(\mathbf{y},\mathbf{u},\mathbf{p}) \;:=\; \mathbf{J}(\mathbf{y},\mathbf{u}) + \langle \mathbf{p}, \mathbf{A}\mathbf{y} - (\mathbf{f}+\mathbf{u})\rangle$$

$\delta\mathbf{Lagr} = 0 \;\rightsquigarrow$

$$\boxed{\begin{aligned} \mathbf{A}\mathbf{y} &= \mathbf{f}+\mathbf{u} \\ \mathbf{A}^T\mathbf{p} &= -\mathbf{y}+\mathbf{y}_* \\ \omega\mathbf{u} &= \mathbf{p} \end{aligned}} \iff \boxed{\mathbf{Q}\mathbf{u} = \mathbf{g}} \quad \text{where} \quad \begin{aligned} \mathbf{Q} &:= \mathbf{A}^{-T}\mathbf{A}^{-1} + \omega\mathbf{I} \\ \mathbf{g} &:= \mathbf{A}^{-T}(-\mathbf{A}^{-1}\mathbf{f} - \mathbf{y}_*) \end{aligned}$$

$\mathbf{Q}$ symmetric positive definite

Convergent Iteration for the ∞-Dimensional Problem

Iterative solution of $\mathbf{M}\mathbf{u} = \mathbf{f}$ $\mathbf{M}$ symmetric positive definite

$\rightsquigarrow$ $\mathbf{u}^{n+1} = \mathbf{u}^n + \alpha\,(\mathbf{f} - \mathbf{M}\mathbf{u}^n)$ $n = 0, 1, 2, \ldots$ $0 < \alpha_* \le \alpha \le \alpha^*$

$\rightsquigarrow$ $\|\mathbf{u}^{n+1} - \mathbf{u}\| \le \rho\,\|\mathbf{u}^n - \mathbf{u}\|$ where $\rho := \|\mathbf{I} - \alpha\mathbf{M}\| < 1$

Ideal iteration

$\rightsquigarrow$ computable scheme for evaluation of $\mathbf{Q}\mathbf{u}^n = (\mathbf{A}^{-T}\mathbf{A}^{-1} + \omega\mathbf{I})\mathbf{u}^n$ and $\mathbf{g}$

Adaptive Approximate Iterations

RES $[\eta, \mathbf{Q}, \mathbf{g}, \mathbf{v}] \to \mathbf{r}_\eta$ DETERMINES FOR GIVEN $\eta > 0$
A FINITELY SUPPORTED $\mathbf{r}_\eta$ SATISFYING $\|\mathbf{g} - \mathbf{Q}\mathbf{v} - \mathbf{r}_\eta\| \leq \eta$

COARSE $[\eta, \mathbf{w}] \to \mathbf{w}_\eta$ DETERMINES FOR GIVEN $\eta > 0$
A FINITELY SUPPORTED $\mathbf{w}_\eta$ SATISFYING $\|\mathbf{w} - \mathbf{w}_\eta\| \leq \eta$

Realization: sort $\mathbf{w}$ into nonincreasing order $\rightsquigarrow \mathbf{w}^*$
find smallest k such that sum of k largest coefficients exceeds $\|\mathbf{w}\|^2 - \eta^2$ $\rightsquigarrow \mathbf{w}_\eta$

Cost: $2N$ and $N \log N$ for sorting if $N = \#(\operatorname{supp} \mathbf{W})$

Main algorithm: $\#$ interior iterations is $K := \min\{\ell : \rho^{\ell-1}(\alpha\ell + \rho) \leq \frac{1}{10}\}$
(α ralaxation weight $\rho < 1$ contraction number)

Adaptive Approximate Iterations – Main Algorithm

SOLVE $[\varepsilon, \mathbf{Q}, \mathbf{g}] \to \mathbf{u}_\varepsilon$

(I) $j = 0$ $\mathbf{u}^0 = \mathbf{0}$ $\varepsilon_0 := \frac{1}{2} c_{\mathbf{A}}^{-1}(c_{\mathbf{A}}^{-1}\|\mathbf{f}\| + \|\mathbf{y}_*\|)$

(II) IF $\varepsilon_j \le \varepsilon$: STOP AND SET $\mathbf{u}_\varepsilon := \mathbf{u}^j$
OTHERWISE $\mathbf{v}^0 := \mathbf{u}^j$

(II.1) FOR $n = 0, \ldots, K-1$ COMPUTE RES $[\rho^n \varepsilon_j, \mathbf{Q}, \mathbf{g}, \mathbf{v}^n] \to \mathbf{r}^n$
AND $\mathbf{v}^{n+1} := \mathbf{v}^n + \alpha\, \mathbf{r}^n$

(II.2) APPLY COARSE $[\frac{2}{5}\varepsilon_j, \mathbf{v}^K] \to \mathbf{u}^{j+1}$ SET $\varepsilon_{j+1} := \frac{1}{2}\varepsilon_j$ $j+1 \to j$
GO TO (II)

Theorem [Cohen, Dahmen, DeVore] For any $\varepsilon > 0$
SOLVE $[\varepsilon, \mathbf{Q}, \mathbf{g}] \to \mathbf{u}_\varepsilon$ terminates after finitely many steps and $\|\mathbf{u} - \mathbf{u}_\varepsilon\| \le \varepsilon$

Routines for Realization of RES

APPLY $[\eta, \mathbf{A}, \mathbf{v}] \to \mathbf{w}_\eta$ COMPUTES FOR GIVEN $\eta > 0$ A FINITELY SUPPORTED $\mathbf{w}_\eta$ SATISFYING $\|\mathbf{A}\mathbf{v} - \mathbf{w}_\eta\| \le \eta$

SOLVE $[\eta, \mathbf{A}, \mathbf{f} + \mathbf{u}] \to \mathbf{y}_\eta$ COMPUTES FOR GIVEN $\eta > 0$ A FINITELY SUPPORTED $\mathbf{y}_\eta$ SATISFYING $\|\mathbf{y} - \mathbf{y}_\eta\| \le \eta$

employs RESELL $[\eta, \mathbf{A}, \mathbf{f} + \mathbf{u}, \mathbf{y}] \to \mathbf{r}_\eta$

(i) APPLY$[\frac{1}{2}\eta, \mathbf{A}, \mathbf{y}] \to \mathbf{w}_\eta$

(ii) COARSE$[\frac{1}{4}\eta, \mathbf{f}] \to \mathbf{f}_\eta$

COARSE$[\frac{1}{4}\eta, \mathbf{u}] \to \mathbf{u}_\eta$

(iii) SET $\mathbf{r}_\eta := \mathbf{f}_\eta + \mathbf{u}_\eta - \mathbf{w}_\eta$

SOLVE $[\eta, \mathbf{A}^T, -\mathbf{y} + \mathbf{y}_*] \to \mathbf{p}_\eta$ COMPUTES FOR GIVEN $\eta > 0$ A FINITELY SUPPORTED $\mathbf{p}_\eta$ SATISFYING $\|\mathbf{p} - \mathbf{p}_\eta\| \le \eta$

Realization of $\textsc{Res}\,[\rho^n \varepsilon_j, \ldots]$ — $(n+1)$th iterate in $(j+1)$th block

$\eta := \rho^n \varepsilon_j$ $\qquad$ $\textsc{Res}\,[\eta, \mathbf{Q}, \mathbf{g}, \mathbf{v}] \longrightarrow \mathbf{r}_\eta$

(I) $\delta_{\mathbf{u}} := \rho^{n-1}\varepsilon_j(\rho + \alpha n)$ $\qquad$ $\delta_{\mathbf{y}} := c_{\mathbf{A}}^{-1}\,\delta_{\mathbf{u}} + \eta$

(II) $\textsc{Coarse}[4\delta_{\mathbf{y}}, \mathbf{y}^{j+1,n}] \longrightarrow \mathbf{y}_\eta^{j+1,n+1,0}$

(III) $\textsc{Solve}[\frac{1}{2}\,c_{\mathbf{A}}\,\eta, \mathbf{A}, \mathbf{f}, \mathbf{u}^{j+1,n}, \mathbf{y}_\eta^{j+1,n+1,0}] \longrightarrow \mathbf{y}_\eta$ $\qquad$ $=: \mathbf{y}^{j+1,n+1}$

(IV) $\textsc{Coarse}[\frac{4}{\omega}\delta_{\mathbf{u}}, \mathbf{p}^{j+1,n}] \longrightarrow \mathbf{p}_\eta^{j+1,n+1,0}$

(V) $\textsc{Solve}[\frac{1}{2}\eta, \mathbf{A}^T, -\mathbf{y}_\eta + \mathbf{y}_*, \mathbf{p}_\eta^{j+1,n+1,0}] \longrightarrow \mathbf{p}_\eta$ $\qquad$ $=: \mathbf{p}^{j+1,n+1}$

(VI) SET $\mathbf{r}_\eta := -\mathbf{p}_\eta - \omega\mathbf{v}$

Convergence and Complexity [Cohen, Dahmen, DeVore]

Goal: Show that SOLVE realizes asymptotically the work/accuracy balance of best N–term approximation

Ideal Bench Mark — Best N-Term Approximation

$$\|\mathbf{v} - \mathbf{v}_N\| := \min_{\#\,\mathrm{supp}\,\mathbf{w} \leq N} \|\mathbf{v} - \mathbf{w}\|$$

$\rightsquigarrow$ classify 'sparse' sequences in ℓ_2 whose best N–term approximation decays at certain rate $\rightsquigarrow$ Lorentz spaces

$$\ell_\tau^w := \{\mathbf{v} \in \ell_2 : \ \#\{\lambda \in I\!I : |v_\lambda| > \eta\} \ \lesssim \ \eta^{-\tau}\} \quad \text{for } 0 < \tau < 2$$

$$|\mathbf{v}|_{\ell_\tau^w} := \sup_{n \in I\!N} n^{1/\tau} v_n^* \qquad (\mathbf{v}^* \text{ decreasing rearrangement})$$

$$\|\mathbf{v}\|_{\ell_\tau^w} := \|\mathbf{v}\| + |\mathbf{v}|_{\ell_\tau^w} \qquad \text{quasi–norm for } \ell_\tau^w$$

$$\ell_\tau \subset \ell_\tau^w \subset \ell_{\tau+\delta} \subset \ell_2 \qquad \text{for } \tau < \tau + \delta < 2 \qquad \ell_\tau^w \text{ 'close' to } \ell_\tau$$

Convergence and Complexity [Cohen, Dahmen, DeVore]

Proposition $\mathbf{v} \in \ell^w_\tau \quad \Longleftrightarrow \quad \|\mathbf{v} - \mathbf{v}_N\| \lesssim N^{-s}\|\mathbf{v}\|_{\ell^w_\tau}$ $\qquad \frac{1}{\tau} = s + \frac{1}{2}$

Ideal work/accuracy rate: target accuracy $\varepsilon \longleftrightarrow$ work $\varepsilon^{-1/s}$

Coarsening Lemma $\mathbf{v} \in \ell^w_\tau$ $\quad \mathbf{w}$ finitely supported such that $\|\mathbf{v} - \mathbf{w}\| \leq \eta$

$\Longrightarrow$ output $\mathbf{w}_\eta$ of $\textsc{Coarse}\,[4\eta, \mathbf{w}]$ satisfies

$$\#\,\mathrm{supp}\,\mathbf{w}_\eta \lesssim \|\mathbf{v}\|^{1/\tau}_{\ell^w_\tau}\,\eta^{-1/s}$$

$$\|\mathbf{v} - \mathbf{w}_\eta\| \lesssim 5\eta$$

$$\|\mathbf{w}_\eta\|_{\ell^w_\tau} \lesssim \|\mathbf{v}\|_{\ell^w_\tau}$$

Convergence and Complexity — Compressible Matrices [CDD]

$\mathbf{A}$ s^*–compressible: for any $0 < s < s^*$ there exists $\mathbf{A}_j$ with $\leq \alpha_j 2^j$ nonzero entries per row and column such that

$$\|\mathbf{A} - \mathbf{A}_j\| \leq \alpha_j 2^{-sj} \qquad j \in I\!N_0 \qquad \sum_{j \in I\!N_0} \alpha_j < \infty$$

$\mathbf{v}$ finitely supported $\rightsquigarrow$ $\mathbf{v}_{[j]} := \mathbf{v}_{2^j}$ best 2^j–aproximations

$$\mathbf{w}_j := \mathbf{A}_j \mathbf{v}_{[0]} + \mathbf{A}_{j-1} \mathbf{v}_{[1]} + \cdots + \mathbf{A}_0 \mathbf{v}_{[j]}$$

Theorem $\mathbf{A}$ s^*–compressible $\rightsquigarrow$

$\mathbf{A}$ bounded on ℓ^w_τ for $s < s^*$ and $\frac{1}{\tau} = s + \frac{1}{2}$

$\mathbf{v}$ finitely supported $\rightsquigarrow$ output $\mathbf{w}_\eta$ of $\text{APPLY}\,[\eta, \mathbf{A}, \mathbf{v}]$ satisfies

$$\|\mathbf{w}_\eta\|_{\ell^w_\tau} \lesssim \|\mathbf{v}\|_{\ell^w_\tau}$$

$$\#\,\mathrm{supp}\,\mathbf{w}_\eta,\ \#\text{flops} \lesssim \eta^{-1/s} \|\mathbf{v}\|^{1/s}_{\ell^w_\tau}$$

Convergence and Complexity — Main Theorem

Lemma $\textsc{Res}\,[\eta, \mathbf{Q}, \mathbf{g}, \mathbf{v}] \to \mathbf{w}_\eta$ satisfies

$$\|\mathbf{w}_\eta\|_{\ell^w_\tau} \lesssim \max\{\|\mathbf{v}\|_{\ell^w_\tau}, \|\mathbf{g}\|_{\ell^w_\tau}\}$$

$$\#\,\mathrm{supp}\,\mathbf{w}_\eta \lesssim \eta^{-1/s} \max\{\|\mathbf{v}\|^{1/s}_{\ell^w_\tau}, \|\mathbf{g}\|^{1/s}_{\ell^w_\tau}\} \qquad \frac{1}{\tau} = s + \frac{1}{2}$$

$$\#\text{flops} \sim \#\,\mathrm{supp}\,\mathbf{w}_\eta$$

Theorem [Cohen, Dahmen, DeVore]

$\textsc{Res}$ has above properties for fixed $\tau^* > 0$ and any $\tau \in (\tau^*, 2)$

$\Longrightarrow$ for every $\varepsilon > 0$ $\textsc{Solve}\,[\varepsilon, \mathbf{Q}, \mathbf{g}] \to \mathbf{u}_\varepsilon$ converges in finitely many steps

$\|\mathbf{u} - \mathbf{u}_\varepsilon\| \le \varepsilon$ $\mathbf{u}_\varepsilon$ is finitely supported

If $\mathbf{u} \in \ell^w_\tau$ for some $\tau \in (\tau^*, 2)$ $\Longrightarrow$

$$\#\,\mathrm{supp}\,\mathbf{u}_\varepsilon \lesssim \varepsilon^{-1/s}\|\mathbf{u}\|^{1/s}_{\ell^w_\tau}$$

$$\|\mathbf{u}_\varepsilon\|_{\ell^w_\tau} \lesssim \|\mathbf{u}\|_{\ell^w_\tau}$$

$$\#\text{flops} \sim \#\,\mathrm{supp}\,\mathbf{u}_\varepsilon$$