Quiz 3 Solutions

Math 1572H, 14 February 2006

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1. [5 points] Find the limit

$$\lim_{x\to 0^+} \sin x \ln x.$$

Solution: In order to apply L'Hospital's rule, we must rewrite the function in an indeterminate form $0/0$ or ∞/∞. We can do this by writing

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}}.$$

Now we can use L'Hospital's rule to conclude that the limit above is equal to

$$\begin{aligned} \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}} \\ &= \lim_{x\to 0^+} \frac{\sin^2 x}{x\cos x}. \end{aligned}$$

Because this in indeterminate of the form $0/0$, we can use L'Hospital again to get

$$\begin{aligned} \lim_{x\to 0^+} \sin x \ln x &= \lim_{x\to 0^+} \frac{\sin^2 x}{x\cos x} \\ &= \lim_{x\to 0^+} \frac{2\sin x\cos x}{x\sin x + \cos x} \\ &= \frac{0}{0+1} \\ &= 0. \end{aligned}$$

2. [5 points] A thin metal plate of uniform thickness is formed by joining a half-disk of radius 1 to the side of a square with side length 2. Describe the location of the center of mass of this plate.

Solution: To do the calculations, we put the plate on a coordinate system so that the corners of the square at at the points $(1,0), (1,-2), (-1,-2)$, and $(-1,0)$. We put the half-disk above the square with center at the origin. That is, the disk is the upper half of a disk with radius 1 and center at the origin.

When set up in this way, symmetry tells us immediately that the x-coordinate of the center of mass is 0.

The y-coordinate is given by

$$\begin{aligned}
\bar{y} &= \frac{1}{A}\frac{1}{2}\int_{-1}^{1}[(\sqrt{1-x^2})^2-(-2)^2]\,dx \\
&= \frac{1}{2A}\int_{-1}^{1}(1-x^2-4)\,dx \\
&= \frac{1}{2A}\int_{-1}^{1}(-3-x^2)\,dx \\
&= \frac{1}{2A}(-3x-\frac{x^3}{3})\Big|_{-1}^{1} \\
&= \frac{1}{2A}(-3-(1/3)-3-(1/3)) \\
&= \frac{-10}{3A}.
\end{aligned}$$

The area A is the area of the square (4 square units) plus the area of the half-disk ($\pi/2$ square units). Therefore,

$$\bar{y} = \frac{-10}{4+(3\pi/2)}$$

in this coordinate system. To describe in words the location of the center of mass, it is midway from one side of the square to the other, and $\frac{10}{4+(3\pi/2)}$ units into the square from the seam with the half-disk.

3. [2 bonus points] Give an example of a Pythagorean Triple involving the number 82. You may use the fact that a parametrization for the unit circle (without the point $(-1,0)$) is given by

$$x(t) = \frac{1-t^2}{1+t^2}, \qquad y(t) = \frac{2t}{1+t^2},$$

where $-\infty < t < \infty$.

POSSIBLE HINT: $82 = 1 + 81$.

Solution: Let $t = 9$ in the parametrization. Then we find that the point $(x(t), y(t)) = (80/82, 18/82)$ is a point on the unit circle with rational coordinates. Because it's on the unit circle, we know that

$$\left(\frac{80}{82}\right)^2 + \left(\frac{18}{82}\right)^2 = 1,$$

which tells us that $80^2 + 18^2 = 82^2$. Therefore, an example of a Pythagorean Triple would be $(18, 80, 82)$.