Quiz 8 Solutions

Math 1572H, 6 April 2006

SHOW ALL YOUR WORK. LABEL YOUR ANSWERS WITH THE PROBLEM NUMBERS.

1. [2 bonus points] What is the Taylor series about 0 of the function f given by

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x^2}} & x > 0? \end{cases}$$

Solution: One can show that when $x > 0$, $f^{(n)}(x)$ is of the form $q_n(x)e^{-1/x^2}$, where $q_n(x)$ is a rational function of x. As x approaches zero from above, the exponential goes to zero faster than the rational function goes to infinity. Therefore,

$$\lim_{x \to 0^+} f^{(n)}(x) = 0 \text{ for all } n \geq 0.$$

Of course, when $x < 0$, all derivatives are exactly zero. This gives us reason to suspect that $f^{(n)}(0)$ exists and equals 0 for all n. Therefore, the Taylor series is

$$T(x) = \sum a_n x^n = \sum 0 \cdot x^n = 0.$$

NOTE: To do this problem properly, we should use the limit definition of the derivative to calculate ensure that the derivative at zero exists and equals zero.

2. [1 bonus point] Give an example of a function $g(x)$ whose Taylor series has radius of convergence $+\infty$, but for all $x > 0$,

$$g(x) \neq \sum_{n=0}^{\infty} a_n x^n.$$

Solution: The function $f(x)$ given in problem 1 works here. Notice that $f(x) > 0$ for all $x > 0$, but $T(x) = 0$ (where $T(x)$ is its Taylor series). The Taylor series is a convergent series, but doesn't converge to the function.

3. [2 bonus points] Find the sum of the series (in terms of λ)

$$\sum_{n=0}^{\infty} n^2 \lambda^n \frac{e^{-\lambda}}{n!}.$$

Solution: This is part of the calculation of the variance of a Poisson random variable. See Worksheet 5 answers for the answer.

4. [2 bonus points] Find the sum of the series (assuming that x is within the interval of convergence)

$$\frac{x^4}{4} + \frac{x^8}{8} + \frac{x^{12}}{12} + \frac{x^{16}}{16} + \cdots.$$

Solution: Let $f(x) = \frac{x^4}{4} + \frac{x^8}{8} + \frac{x^{12}}{12} + \frac{x^{16}}{16} + \cdots$.. Then

$$\begin{aligned} f'(x) &= x^3 + x^5 + x^{11} + x^{15} + \ldots \\ &= \sum_{n=1}^{\infty} x^{4n-1} \\ &= \sum_{n=0}^{\infty} x^{4n+3} \\ &= x^3 \sum_{n=0}^{\infty} x^{4n} \\ &= x^3 \frac{1}{1-x^4}. \end{aligned}$$

Since $f'(x) = x^3/(1-x^4)$, we integrate to find that $f(x) = (-1/4)\ln(1-x^4) + C$. Using the fact that the sum is clearly 0 when $x = 0$, we find that $C = 0$.

5. [1 bonus point] What is the first letter of the capital of the state where Ryan was born? M
What is the second letter of the middle name of president (of the U.S.) when Ryan was born? A
What is the third letter of the name of the city which is the capital of Texas? S
What is the last letter in the word ‘math’? H

Solution: Madison, Earl, Austin, Math.