Worksheet 2: Solutions to In-Class Problems

Math 1572H, 30 January 2006

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1. Evaluate the integral

$$\int \frac{dx}{e^{3x} - e^x}.$$

Solution: Let $u = e^x$. Then $du = e^x\,dx = u\,dx$, so that $dx = du/u$. We translate the integral from the language of x to the language of u and get

$$\int \frac{dx}{e^{3x} - e^x} = \int \frac{1}{u^3 - u}\frac{du}{u}.$$

Now we can rewrite the integrand using partial fractions. With the setup

$$\frac{1}{u^2(u+1)(u-1)} = \frac{Au + B}{u^2} + \frac{C}{u+1} + \frac{D}{u-1},$$

we deduce algebraically (I'll leave this to you) that $A = 0$, $B = -1$, $C = -1/2$, and $D = 1/2$. Therefore, the integral becomes

$$\int \left(\frac{-1}{u^2} - \frac{1/2}{u+1} + \frac{1/2}{u-1} \right) du.$$

We do the integration and plug x back in to get an answer of

$$e^{-x} - \frac{1}{2}\ln(e^x + 1) + \frac{1}{2}\ln(e^x - 1) + C.$$

You should take the derivative of this answer to check it.

2. Evaluate

$$\int x \sin x \cos x\,dx.$$

Solution: We'll start with integration by parts. We let $u = x \sin x$, leaving $dv = \cos x\,dx$. We find that $du = (x \cos x + \sin x)\,dx$ and $v = \sin x$. Using the integration by parts formula, the integral is equal to

$$x \sin^2 x - \int (x \cos x + \sin x) \sin x\,dx.$$

We factor out the product under the integral sign at the right of the last line and write it as two integrals to obtain the expression

$$\int x \sin x \cos x\,dx = x \sin^2 x - \int \sin^2\ dx - \int x \sin x \cos x\,dx. \tag{1}$$

The trick that is involved is to realize that there is a common expression on both sides of line (1), and that this expression is exactly what we're looking for. By adding $\int x \sin x \cos x\,dx$ to both sides of (1), we get

$$2\int x \sin x \cos x\,dx = x \sin^2 x - \int \sin^2 x\,dx.$$

Now we simply divide by two and integrate $\sin^2 x$ using a half-angle formula to get an answer. I'll leave the details to you, but I get an answer of

$$\frac{1}{2}x \sin^2 x - \frac{1}{4}x + \frac{1}{4}\sin x \cos x + C.$$

Again, you should take the derivative to be sure that you're right.

3. Evaluate

$$\int \cos x \left(\ln(\sin x)\right)\, dx.$$

Solution: My solution involves both substitution and integration by parts. Since the letter u is typically used for both, I'll try to minimize confusion (and maximize fun!) by using y for the substitution instead. Yeah, I'm a rebel.

Let $y = \sin x$. Then $dy = \cos x\, dx$ and the integral is simply

$$\int \ln(y)\, dy.$$

To integrate the logarithm function, we integrate by parts. Let $u = \ln y$ and $dv = dy$. Then $du = (1/y)\, dy$ and $v = y$. Using the formula we find

$$\int \ln y\, dy = y \ln y - \int dy = y \ln y - y + C.$$

The answer is, in terms of x, equal to

$$\sin x \ln(\sin x) - \sin x + C.$$

4. Evaluate

$$\int \frac{x}{1 - x^2 + \sqrt{1 - x^2}}\, dx.$$

Solution: The way I did this problem involved two substitutions. Of course, whenever this is the case, it could probably be done with one, more clever substitution. Therefore, I'll give you the substitutions that I used and let you work the problem from there. Here are my hints:

- Let $y = 1 - x^2$.
- Now let $z = \sqrt{y}$.
- Convince yourself that $2z\, dz = dy$.
- Plug this into the integral, simplify the fraction.
- Take the derivative of your answer to check it.