(*October 3, 2023*)

Examples 02

Paul Garrett garrett@umn.edu https://www-users.cse.umn.edu/~garrett/

[02.1] Find all the idempotent elements in $\mathbb{Z}[i]/\langle 13\rangle$.

[02.2] Find all the nilpotent elements in $\mathbb{Z}[i]/\langle 2\rangle$.

[02.3] *(Lagrange interpolation)* Let $\alpha_1, \ldots, \alpha_n$ be *distinct* elements in a field k, and let $\beta_1, \ldots, \beta_n$ be any elements of k. Prove that there is a unique polynomial $P(x)$ of degree $< n$ in $k[x]$ such that, for all indices i,

$$P(\alpha_i) = \beta_i$$

Indeed, letting

$$Q(x) = \prod_{i=1}^{n} (x - \alpha_i)$$

show that

$$P(x) = \sum_{i=1}^{n} \frac{Q(x)}{(x - \alpha_i) \cdot Q'(\alpha_i)} \cdot \beta_i$$

[02.4] *(Simple case of partial fractions)* Let $\alpha_1, \ldots, \alpha_n$ be *distinct* elements in a field k. Let $R(x)$ be any polynomial in $k[x]$ of degree $< n$. Show that there exist unique constants $c_i \in k$ such that in the field of rational functions $k(x)$

$$\frac{R(x)}{(x - \alpha_1) \ldots (x - \alpha_n)} = \frac{c_1}{x - \alpha_1} + \ldots + \frac{c_n}{x - \alpha_n}$$

In particular, let

$$Q(x) = \prod_{i=1}^{n} (x - \alpha_i)$$

and show that

$$c_i = \frac{R(\alpha_i)}{Q'(\alpha_i)}$$

[02.5] *(Analogue of partial fractions for rational numbers)* Show that every positive rational number is expressible as

$$\ell + \sum_p \frac{c_p}{p^{n_p}} \qquad (0 \le \ell \in \mathbb{Z}, \text{ distinct primes } p, \text{ exponents } 0 \le n_p \in \mathbb{Z}, \text{ integers } 0 \le c_p < p^{n_p})$$

[02.6] Show that the ideal I generated in $\mathbb{Z}[x]$ by $x^2 + 1$ and 5 is *not* maximal.

[02.7] Show that the ideal I generated in $\mathbb{Z}[x]$ by $x^2 + x + 1$ and 11 is maximal.

[02.8] Let k be a field. Given $P \in k[x]$ of degree n, show that there is a k-linear map $T : k^n \to k^n$ such that $P(T) = 0$.

[02.9] Determine all two-sided ideals in the ring of n-by-n matrices with entries in a field k.

[02.10] Let $V_1 \subset \ldots \subset V_{n-1} \subset V$ and $W_1 \subset \ldots \subset W_{n-1} \subset V$ be two maximal flags in an n-dimensional vector space V over a field k. Show that there is a k-linear map $T : V \to V$ such that $TV_i = W_i$.
