Solution 5

#1 Let G be a group of prime order. Show that G is cyclic.

Let g be any element other than the identity in G. We will show that $\langle g \rangle = G$. By Lagrange's theorem, any subgroup of G has order dividing p, so has order either 1, or p. The subgroup $\langle g \rangle$ contains at least two elements, namely e and g (which are distinct because $g \neq e$). Thus, $|\langle g \rangle| = p$. Thus, this subgroup is the whole group.

#2 Show that in an abelian group every subgroup is normal.

Let G be an abelian group, and N a subgroup. Let $g \in G$. Then

$$gNg^{-1} = \{gng^{-1} : n \in N\} = \{gg^{-1}n : n \in N\} = \{n : n \in N\} = N$$

Thus, N is normal.

#3 Show that any abelian group of order 21 is cyclic.

By Cauchy's theorem, there are elements x, y of orders 3 and 7, respectively. We claim that xy is of order 21. First, of course, we can note that $\langle x \rangle \cap \langle y \rangle = \{e\}$, since (by Lagrange) the order of the intersection must divide both $|\langle x \rangle = 3$ and $|\langle y \rangle = 7$, so must be 1. By Lagrange's theorem, the order of xy is in the list $1, 3, 7, 21$. If $xy = e$, then $x = y^{-1}$, and the latter element lies in $\langle x \rangle \cap \langle y \rangle = \{e\}$. This would imply that $x = y^{-1} = e$, which is not the case, so the order of xy is not 1. Suppose that $(xy)^3 = e$. Since the group is abelian, this would imply that $x^3y^3 = e$, so $e = x^3 = y^{-3}$ since $x^3 = e$. But $y^7 = e$ by hypothesis, so that if also $y^{-3} = e$ then $y = y^7 \cdot (y^{-3})^2 = e$, contradiction. So the order of xy is not 3. Symmetrically, its order is not 7, so by default it is 21. That is xy generates the group, so the group is cyclic.