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Asymptotics at irregular singular points

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According to [Erdélyi 1956], Thomé [1] found that differential equations with *finite rank* irregular singular points have asymptotic expansions given by the expected recursions. Thus, although the irregularity typically precludes *convergence* of the series expression for solutions, the series is still a legitimate *asymptotic expansion*.

We approximately follow [Erdélyi 1956] in treating a rank-one irregular singular point of a second-order differential equation: after normalization to get rid of the first-derivative term, these are of the form

$$u'' - q(x)\,u \;=\; 0 \qquad \text{with} \qquad q(x) \;\sim\; q_o + \frac{q_1}{x} + \frac{q_2}{x^2} + \ldots \qquad (\text{as } x \to +\infty, \text{ with } q_o \neq 0)$$

with q continuous in some range $x \geq a$. The series expression for $q(x)$ need not be convergent: it suffices that it be an *asymptotic* expansion of $q(x)$ at $+\infty$. *Freezing* the coefficient q to its value at $+\infty$, gives the *constant-coefficient* equation

$$u'' - q_o\,u \;=\; 0$$

and *suggests* that the solutions $e^{\pm\sqrt{q_o}\,x}$ of the constant-coefficient equation should give the leading term in the asymptotics of solutions of the original equation. This is approximately true: there is an adjustment by a power of x. Solutions have asymptotics of the form

$$u(x) \;\sim\; e^{\pm\sqrt{q_o}\,x} \cdot x^\rho \cdot \Big(1 + \sum_{n\geq 1} \frac{a_n}{x^n}\Big) \qquad (\text{with } \rho = \frac{q_1}{\pm 2\sqrt{q_o}}, \text{ as } x \to +\infty)$$

with coefficients a_n obtained by a natural recursion. However, the series rarely converges.

The loss of convergence is not a trifling matter. The term-wise differentiability of convergent power series is extremely useful. In contrast, term-wise differentiation of *asymptotic* series

$$f(x) \;\sim\; \sum_{n\geq 0} \frac{a_n}{x^n} \qquad (\text{as } x \to +\infty)$$

for differentiable f produces an asymptotic series for f' only under additional hypotheses, for example, that f' admits such an asymptotic series. (See the appendix.) While a function admitting an asymptotic expansion of this form determines that expansion uniquely, the expansion does not uniquely determine the function. For example, as $x \to +\infty$, $e^{-x} = o(x^{-N})$ for all N, so e^{-x} has the 0 asymptotic expansion, but is not the 0 function.

[Olver 1954a] notes that Carlini, [Green 1837], and [Liouville 1837] investigated relatively simple cases, without complete rigor. [Olver 1954a] further notes the work of [Horn 1899], [Schlesinger 1907],

[1] Probably this is Wilhelm Ludwig Thomé, who, according to [Genealogy 2011], wrote a thesis *De seriebus secundum functiones, quae vocantur sphaericae progredientibus* in 1865 at Berlin under K. Weierstraß.

[Birkhoff 1908], [Tamarkin 1928], [Turritin 1936] on asymptotics on intervals free from *transition points*, that is, points where coefficient functions vanish or have singularities, thus changing the qualitative nature of the equation. Our hypothesis $q(\infty) = q_o \neq 0$ and on the asymptotic expansion of q at $+\infty$ assures that in some neighborhood of $+\infty$ our examples have *no* transition points.

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1. *Example: rotationally symmetric eigenfunctions on* $\mathbb{R}^n$

[1.1] Rotation-invariant eigenfunctions for Δ on $\mathbb{R}^n$

A natural example arises from the eigenvalue equation for the radial component of the Euclidean Laplacian on $\mathbb{R}^n$:

$$v'' + \frac{n-1}{r} v' - \lambda v \;=\; 0$$

For large r, this equation resembles the constant-coefficient equation $v'' - \lambda v = 0$, with solutions $e^{\pm r\sqrt{\lambda}}$. Heuristically, we should have solutions with behavior $v \sim e^{\pm r\sqrt{\lambda}}$ as $v \to +\infty$. This is not quite right: the true asymptotic expansions have main terms

$$v \;\sim\; \frac{e^{\pm r\sqrt{\lambda}}}{r^{\frac{n-1}{2}}}$$

That is, the differences between the actual equation and the constant-coefficient approximation do not alter the constant in the exponential, but do have a significant impact, as we see below.

A natural recursion, carried out just below, produces an apparent solution to differential equations in this class, of the form

$$e^{\omega x}\, x^{-\rho} \sum_{n\geq 0} \frac{c_n}{x^n}$$

However, unlike the regular singular point situation, the series is *not convergent*! The relation of this non-convergent series to any genuine solution is *a priori* unclear. It is natural to suppose that this non-convergent series is an *asymptotic expansion*, but this is not obvious. A genuine solution must be identified by other means, must be proven to *have* an asymptotic expansion, and the latter must be compared with the series obtained by the recursion. All this will occupy us in following sections.

[1.2] Recursion

In more detail, the heuristic recursion is as follows, as applied to the eigenvalue equation for the radial component of the Laplacian on $\mathbb{R}^n$. First, simplify by employing the standard device to eliminate the v' term: [2] take $v = u/r^{(n-1)/2}$, and then

$$u'' - \Big(\lambda + \frac{(n-1)(n-3)}{4\,r^2}\Big)\, u \;=\; 0$$

The singular point at infinity is *irregular*, unless $n = 1$ or 3. Nevertheless, intuitively, for $x \to +\infty$, a differential equation of the shape

$$u'' - (\lambda + \frac{C}{x^2})u \;=\; 0$$

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[2] Let $v = u \cdot w$ and set the u' term equal to 0 in the left-hand side. This gives $2u'w' + \frac{n-1}{r}u'w = 0$, which gives the differential equation $2w' + \frac{n-1}{r}w = 0$ for w.

is approximately a constant-coefficient differential equation, suggesting a solution of the form [3]

$$u(x) \;=\; e^{\pm x\sqrt{\lambda}} \cdot \sum_{\ell=0}^{+\infty} \frac{c_\ell}{x^\ell} \qquad\qquad \text{(with } c_0 = 1\text{, without loss of generality)}$$

Substituting the latter into the differential equation and dividing through by $e^{\pm x\sqrt{\lambda}}$, letting $s = \pm\sqrt{\lambda}$, simplifies to

$$\sum_{\ell=0}^{+\infty} \Big((\ell-2)(\ell-1)c_{\ell-2} - 2s(\ell-1)c_{\ell-1}\Big)\frac{1}{x^\ell} \;-\; \sum_{\ell=0}^{+\infty} c_{\ell-2}\frac{C}{x^\ell} \;=\; 0$$

where by convention $c_{-1} = c_{-2} = 0$. The case $\ell = 0$ is vacuous, as is $\ell = 1$. The case $\ell \geq 2$ determines $c_{\ell-1}$, assuming $s \neq 0$:

$$(\ell-2)(\ell-1)c_{\ell-2} - 2s(\ell-1)c_{\ell-1} - C\cdot c_{\ell-2} \;=\; 0$$

or

$$c_{\ell+1} \;=\; \frac{\ell(\ell+1) - C}{2s(\ell+1)} \cdot c_\ell$$

[1.2.1] Remark: That recursion causes the coefficients to grow approximately as factorials, and the resulting series *does not converge* for any finite non-zero value of $1/x$, unless the constant C happens to be of the form $(\ell-1)(\ell-2)$ for some positive integer ℓ, in which case the series *terminates*, and *is* convergent.

Our later discussion will show that the above recursion *does* correctly determine *asymptotic expansions* for solutions. In particular, the leading part of the asymptotic is

$$v \;=\; \frac{e^{\pm r\sqrt{\lambda}}}{r^{\frac{n-1}{2}}} \cdot \big(1 + O(\frac{1}{r})\big) \qquad\qquad \text{(as } r \to +\infty\text{, in } \mathbb{R}^n\text{)}$$

The denominator $r^{(n-1)/2}$ might be hard to anticipate. [4] Further, this is an *asymptotic* and not merely a *bound*.

In fact, for n odd, the asymptotic is *finite*: the recursion for coefficients terminates, so gives a convergent series: we obtain not merely *asymptotics*, but *equalities*. Thus, in odd-dimensional $\mathbb{R}^n$ the solutions to the differential equation for rotationally-invariant λ-eigenfunctions have elementary expressions. For example,

$$\begin{cases} v \;=\; e^{\pm r\sqrt{\lambda}} & \text{(on } \mathbb{R}^1\text{)} \\[2ex] v \;=\; \dfrac{e^{\pm r\sqrt{\lambda}}}{r} & \text{(on } \mathbb{R}^3\text{)} \\[2ex] v \;=\; e^{\pm r\sqrt{\lambda}}\Big(\dfrac{1}{r^2} - \dfrac{1}{\pm r^3\sqrt{\lambda}}\Big) & \text{(on } \mathbb{R}^5\text{)} \\[2ex] v \;=\; e^{\pm r\sqrt{\lambda}}\Big(\dfrac{1}{r^3} - \dfrac{3}{\pm r^5\sqrt{\lambda}} + \dfrac{3}{r^7\,\lambda}\Big) & \text{(on } \mathbb{R}^7\text{)} \end{cases}$$

The first two cases, $\mathbb{R}^1$ and $\mathbb{R}^3$, are well-known, but the general pattern less so. [5]

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[3] Anticipating the adjustment by x^ρ in general, with ρ determined by the asymptotics $q_o + \frac{q_1}{x} + \ldots$ of the coefficient of u by $\rho = q_1/2\sqrt{q_o}$, in the present example we are fortunate that $q_1 = 0$, so the idea of *freezing* is exactly right.

[4] The symmetry $r \to -r$ imposes a further requirement, and for $\sqrt{\lambda}$ not purely imaginary one of the two solutions swamps the other. Indeed, for $\sqrt{\lambda}$ not purely imaginary, the asymptotic components of the large solution are all larger than the main part of the smaller solution.

[5] The differential equation at hand is a *Bessel equation*: for certain parameter values, Bessel functions are elementary. In fact, in odd dimensions, Fourier transform methods and residue integration tricks yield elementary expressions for many eigenfunctions on $\mathbb{R}^n - 0$, at least among *tempered distributions*.

[1.2.2] Remark: The same technique applies to differential equations

$$u'' - q(x)\,u \;=\; 0$$

with $q(x)$ continuous in some range $x > a$ and admitting an *asymptotic expansion* at infinity of the form

$$q(x) \;\sim\; \sum_{\ell\geq 0} \frac{q_\ell}{x^\ell} \qquad\qquad (\text{with } q_o \neq 0)$$

The condition $q_o \neq 0$ is essential[6] for the recursion to succeed. Adjustment by x^ρ with $\rho = q_1/2\sqrt{q_o}$ would be found necessary when $q_1 \neq 0$. In any case, the recursion *rarely* produces a convergent power series!

[1.3] Comparison to regular singular points

The behavior of the above recursion is is much different from that resulting from a regular singular point. A power series in $z = 1/x$ behaves differently under d/dx than under d/dz. Indeed, as in the example above, the power series in $1/x$ often diverges, while at a regular singular point the analogous power series has a positive radius of convergence. For $u'' - q(x)u = 0$ to have a regular singular point at infinity, changing variables to $u(x) = v(1/x)$ and $z = 1/x$,

$$u'(x) \;=\; \frac{-1}{x^2} v'(1/x) \qquad\text{and}\qquad u''(x) \;=\; \frac{1}{x^4} v''(1/x) + \frac{2}{x^3} v'(1/x)$$

Putting $z = 1/x$, this is

$$u' \;=\; -z^2 v' \qquad\text{and}\qquad u'' \;=\; z^4 v'' + 2z^3 v' \qquad\qquad (\text{with } u = u(x),\, v = v(z),\, z = 1/x)$$

Thus, in the coordinate z at infinity, the differential equation becomes

$$\Big(z^4 v'' + 2z^3 v'\Big) - q\big(\frac{1}{z}\big)\, v \;=\; 0$$

or

$$v'' + \frac{2}{z} v' - \frac{q(1/z)}{z^4}\, v \;=\; 0$$

The point $z = 0$ is *never* an *ordinary point*, because of the pole in the coefficient of v'. The point $z = 0$ is a regular singular point only when $q(1/z)/z^2$ is analytic at $z = 0$, that is, when $x^2 q(x)$ is analytic at ∞. This requires that $q(x)$ have the form

$$q(x) \;=\; \frac{q_2}{x^2} + \frac{q_3}{x^3} + \ldots$$

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2. *Example: translation-equivariant eigenfunctions on $\mathfrak{H}$*

Another example of irregular singular point arises from the non-Euclidean geometry on the upper half-plane. Recall the $SL_2(\mathbb{R})$-invariant Laplacian on the upper half-plane $\mathfrak{H}$:

$$\Delta^{\mathfrak{H}} \;=\; y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)$$

[6] The condition $q_o \neq 0$ and the assumption that q has the indicated asymptotic at $+\infty$ together imply that there is x_o such that $q(x) \neq 0$ for $x \geq x_o$. That is, in the regime $x \geq x_o$ there are no *transition points*.

[2.1] The ordinary differential equation

We ask for $\Delta^{\mathfrak{H}}$-eigenfunctions $f(z)$ of the special form

$$f(x+iy) \;=\; e^{2\pi i x} u(y)$$

that is, *equivariant* under *translations*:

$$f(z+t) \;=\; e^{2\pi i(x+t)} u(y) \;=\; e^{2\pi i t}\cdot \big(e^{2\pi i x} u(y)\big) \;=\; e^{2\pi i t}\cdot f(z) \qquad\qquad (\text{with } t\in\mathbb{R} \text{ and } z\in\mathfrak{H})$$

The eigenfunction condition

$$(\Delta^{\mathfrak{H}} - \lambda)\, e^{2\pi i x} u(y) \;=\; 0$$

simplifies to the ordinary differential equation

$$y^2 u'' - \Big(4\pi^2 y^2 + \lambda\Big) u \;=\; 0$$

This equation has an *irregular* singular point at $+\infty$, seen by changing coordinates, as follows. Let $u(x) = v(1/x)$ and put $z = 1/x$, obtaining

$$\Big(z^4 v'' + 2z^3 v'\Big) - (4\pi^2 + \lambda z^2) v \;=\; 0$$

or

$$z^2 v'' + 2z v' - \Big(\frac{4\pi^2}{z^2} + \lambda\Big) v \;=\; 0$$

Since the coefficient of v has a pole at $z=0$, the singular point of this equation in the new coordinate z at 0 is *irregular*.

[2.2] Recursion

Happily, following our present prescription, in the form

$$u'' - \Big(4\pi^2 + \frac{\lambda}{y^2}\Big) u \;=\; 0$$

the coefficient

$$q(y) \;=\; q_o + \frac{q_1}{y} + \frac{q_2}{y^2} + \ldots \;=\; 4\pi^2 + \frac{\lambda}{y^2}$$

is analytic at $y=\infty$, and $q(\infty) = q_o = 4\pi^2 \neq 0$ while $q_1 = 0$, so our later discussion justifies *freezing* y at $+\infty$, obtaining the constant-coefficient equation

$$u'' - 4\pi^2 u \;=\; 0$$

with solutions $e^{\pm 2\pi y}$, and assuring existence of solutions of the original equation with asymptotics of the form

$$u(y) \;=\; e^{\pm 2\pi y}\cdot \sum_{\ell\geq 0} \frac{a_\ell}{y^\ell}$$

Substituting this into the differential equation and dividing through by $e^{\pm 2\pi y}$ gives

$$\sum_{\ell=0}^{+\infty} \Big((\ell-2)(\ell-1)a_{\ell-2} \mp 2\pi(\ell-1)a_{\ell-1}\Big)\frac{1}{y^\ell} \;-\; \sum_{\ell=0}^{+\infty} a_{\ell-2}\frac{\lambda}{y^\ell} \;=\; 0$$

or

$$\pm 2\pi(\ell-1)a_{\ell-1} \;=\; \Big((\ell-2)(\ell-1) - \lambda\Big) a_{\ell-2}$$

or

$$a_\ell \;=\; \frac{(\ell-1)\ell-\lambda}{\pm 2\pi\ell}\, a_{\ell-1} \;=\; \Big(\ell-1-\frac{\lambda}{\ell}\Big)\frac{a_\ell}{\pm 2\pi}$$

As usual, $a_{-2}=a_{-1}=0$ by convention, and $a_0=1$. The cases $\ell\le 0$ are vacuous. With $a_0=1$, the recursion begins

$$\begin{aligned} a_1 &= \frac{-\lambda}{\pm 2\pi} \\ a_2 &= \Big(1-\frac{\lambda}{2}\Big)\frac{a_1}{\pm 2\pi} = \Big(1-\frac{\lambda}{2}\Big)\Big(-\lambda\Big)\frac{1}{(\pm 2\pi)^2} \\ a_3 &= \Big(2-\frac{\lambda}{3}\Big)\frac{a_2}{\pm 2\pi} = \Big(2-\frac{\lambda}{3}\Big)\Big(1-\frac{\lambda}{2}\Big)\Big(-\lambda\Big)\frac{1}{(\pm 2\pi)^3} \end{aligned}$$

[2.2.1] Remark: If λ is of the form $\lambda=\ell(\ell-1)$ for $0<\ell\in\mathbb{Z}$, the recursion *terminates*. Then the asymptotic expansion is *convergent*, and produces an elementary solution to the eigenfunction equation. [7]

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3. *Beginning of construction of solutions*

According to [Erdélyi 1956] p. 64, there are roughly two proofs that the above argument produces genuine asymptotic expansions for solutions of the differential equation. Poincaré's approach, elaborated by J. Horn, expresses solutions as Laplace transforms and invokes Watson's lemma to obtain asymptotics. G.D. Birkhoff and his students constructed auxiliary differential equations from partial sums of the asymptotic expansion, and compared these auxiliary equations to the original. Volterra integral operators are important in both approaches, insofar as asymptotic expansions behave better under *integration* than under *differentiation*. The following version of the Birkhoff argument is largely adapted from [Erdélyi 1956], which refers to [Hoheisel 1924] and [Tricomi 1953].

[3.1] Heuristic for asymptotic expansion

Consider the equation

$$u''-q(x)\,u \;=\; 0$$

as $x\to+\infty$, where q is continuous in some range $x>a$ and itself admits an asymptotic expansion

$$q(x)\;\sim\;\sum_{n\ge 0}\frac{q_n}{x^n} \qquad\qquad (\text{as } x\to+\infty, \text{ with } q_o\neq 0)$$

The $q_o\neq 0$ condition is essential. We look for a solution of the form

$$u(x)\;\sim\;e^{\omega x}\cdot x^{-\rho}\cdot\sum_{n\ge 0}\frac{c_o}{x^n} \qquad\qquad (\text{with } c_o \text{ non-zero})$$

Substituting this expansion in the differential equation and dividing through by $e^{\omega x}\,x^{-\rho}$, setting the coefficient of $1/x^n$ to 0,

$$\Big((\rho+n-2)(\rho+n-1)c_{n-2}-2\omega(\rho+n-1)c_{n-1}+\omega^2 c_n\Big)-\Big(q_oc_n+q_1c_{n-1}+\ldots+q_{n-1}c_1+q_nc_o\Big) \;=\; 0$$

By convention, $c_{-2}=c_{-1}=0$ and $q_{-2}=q_{-1}=0$. For $n=0$, the relation is

$$\omega^2 c_o - q_oc_o \;=\; 0$$

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[7] These elementary solutions arise from the *finite-dimensional* representations of $SL_2(\mathbb{R})$.

so $\omega = \pm\sqrt{q_o} \neq 0$, since $c_o \neq 0$. For $n = 1$,

$$\Big(-2\omega\rho c_o + \omega^2 c_1 \Big) - \Big(q_o c_1 + q_1 c_o \Big) \;=\; 0$$

so, using $\omega^2 = q_o$ and $\omega \neq 0$, this is

$$-2\omega\rho - q_1 \;=\; 0$$

so $\rho = -q_1/(2\omega)$. Thus, the choice of $\pm\omega$ is reflected in the choice of $\pm\rho$. For $n \geq 2$, using $\omega^2 = q_o$,

$$\Big(-2\omega(\rho+n-1) - q_1 \Big) c_{n-1} \;=\; -(\rho+n-2)(\rho+n-1)c_{n-2} + \Big(q_2 c_{n-2} + \ldots + q_{n-1}c_1 + q_n c_o \Big)$$

and using $-2\omega\rho - q_1 = 0$,

$$-2\omega(n-1)c_{n-1} \;=\; -(\rho+n-2)(\rho+n-1)c_{n-2} + \Big(q_2 c_{n-2} + \ldots + q_{n-1}c_1 + q_n c_o \Big)$$

Since $\omega \neq 0$, this gives a successful recursion. The following discussion will show that the two asymptotics, with $\pm\omega$ and corresponding $\pm\rho$, are asymptotic expansions of two solutions of the differential equation $u'' - q(x)u = 0$.

[3.1.1] Remark: However, since the above expansions usually do not converge, *genuine* solutions must be constructed by other means, and must be shown to *have* asymptotic expansions at $+\infty$.

[3.2] Small renormalization

For a solution u to $u'' - q(x)u = 0$, let

$$u(x) \;=\; e^{\omega x} \cdot x^{-\rho} \cdot v(x)$$

with ω and ρ determined as above. Then

$$\begin{cases} u' &=\; e^{\omega x} x^{-\rho} \Big((\omega - \frac{\rho}{x}) v + v' \Big) \\ u'' &=\; e^{\omega x} x^{-\rho} \big(\omega - \frac{\rho}{x}\big)^2 v \;+\; e^{\omega x} x^{-\rho} \frac{\rho}{x^2} v \;+\; 2e^{\omega x} x^{-\rho} \big(\omega - \frac{\rho}{x}\big) v' \;+\; e^{\omega x} x^{-\rho} v'' \end{cases}$$

Dividing through by $e^{\omega x} x^{-\rho}$ gives the differential equation for v, namely,

$$v'' + 2\big(\omega - \frac{\rho}{x}\big) v' + \Big(\omega^2 - \frac{2\omega\rho}{x} + \frac{\rho^2+\rho}{x^2} - q(x) \Big) v \;=\; 0$$

Unsurprisingly, the ω^2 and $-2\omega\rho/x$ cancel the first two terms of $q(x)$. Thus, the function

$$F(x) \;=\; x^2 \cdot \Big(\omega^2 - \frac{2\omega\rho}{x} + \frac{\rho^2+\rho}{x^2} - q(x) \Big)$$

is *bounded*. The differential equation is

$$v'' + 2\big(\omega - \frac{\rho}{x}\big) v' + \frac{F(x)}{x^2} v \;=\; 0$$

Rewrite the equation as

$$\frac{d}{dx} \Big(e^{2\omega x} x^{-2\rho} \frac{dv}{dx} \Big) \;+\; e^{2\omega x} x^{-2\rho-2} F(x)\, v(x) \;=\; 0$$

Integrate this from $b \geq a$ to $x \geq b$, and multiply through by $e^{-2\omega x} x^{2\rho}$, to obtain

$$\frac{dv}{dx} + e^{-2\omega x} x^{2\rho} \int_b^x e^{2\omega t}\, t^{-2\rho-2}\, F(t)\, v(t)\, dt \;=\; \text{const} \cdot e^{-2\omega x} x^{2\rho}$$

Take the constant of integration to be 0 and integrate from a to x, to obtain

$$v(x) + \int_a^x e^{-2\omega s}\, s^{2\rho} \Big(\int_b^s e^{2\omega t}\, t^{-2\rho-2}\, F(t)\, v(t)\, dt \Big) ds \;=\; \text{const}$$

Rearrange the double integral:

$$\int_a^x e^{-2\omega s}\, s^{2\rho} \Big(\int_b^s e^{2\omega t}\, t^{-2\rho-2}\, F(t)\, v(t)\, dt \Big) ds \;=\; \int_b^x \Big(\int_t^x e^{2\omega(t-s)} \left(\frac{s}{t}\right)^{2\rho} ds \Big)\, F(t)\, v(t)\, \frac{dt}{t^2}$$

Let $K(x,t)$ denote the inner integral

$$K(x,t) \;=\; \int_t^x e^{2\omega(t-s)} \left(\frac{s}{t}\right)^{2\rho} ds$$

Then the equation is

$$v(x) \;-\; \int_b^x K(x,t)\, F(t)\, v(t)\, \frac{dt}{t^2} \;=\; \text{const}$$

Take the constant to be 1. With $b = +\infty$, this gives an integral equation

$$v(x) \;=\; 1 + \int_x^\infty K(x,t)\, F(t)\, v(t)\, \frac{dt}{t^2}$$

We claim that this equation can be solved by *successive approximations*. With the obvious operator

$$Tf(x) \;=\; \int_x^\infty K(x,t)\, F(t)\, f(t)\, \frac{dt}{t^2}$$

take $w_o(x) = 1$, $w_{n+1} = Tw_n$, and then show that the limit

$$v(x) \;=\; w_o(x) + w_1(x) + w_2(x) + \ldots \;=\; \Big(1 + T + T^2 + \ldots) w_o$$

exists *pointwise*, is *twice differentiable*, and satisfies the differential equation.

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4. *Kernel* $K(x,t)$ *is bounded*

We claim that, with correct choice of $\pm\omega$, the kernel

$$-K(x,t) \;=\; \int_x^t e^{2\omega(t-s)} \left(\frac{s}{t}\right)^{2\rho} ds$$

is bounded for $t \geq x \geq a$. Choose $\pm\omega$ so that either $\mathrm{Re}(\omega) < 0$, or $\mathrm{Re}(\omega) = 0$ and $\mathrm{Re}(\rho) \geq 0$.

[4.1] Very easy case $\rho = 0$

To illustrate the reasonableness of the boundedness assertion, consider the special case $\rho = 0$, where the integral can be computed explicitly:

$$-K(x,t) \;=\; \int_x^t e^{2\omega(t-s)}\, ds \;=\; \frac{1}{-2\omega}\Big(1 - e^{2\omega(t-x)}\Big)$$

Since $\mathrm{Re}\,\omega \leq 0$ and $\omega \neq 0$, this is bounded, for $a \leq x \leq t$.

[4.2] Easy case $\mathrm{Re}\,\omega < 0$

When $\mathrm{Re}\,\omega < 0$, absolute value estimates suffice to prove boundedness of $K(x,t)$.

$$|K(x,t)| \;\le\; \int_x^t e^{2\,\mathrm{Re}\,\omega(t-s)} \Big(\frac{s}{t}\Big)^{2\,\mathrm{Re}\,\rho} ds \;\le\; \int_a^t e^{2\,\mathrm{Re}\,\omega(t-s)} \Big(\frac{s}{t}\Big)^{2\,\mathrm{Re}\,\rho} ds$$

Lighten the notation by taking ω,ρ real. For $\rho \ge 0$,

$$\int_a^t e^{2\omega(t-s)} \Big(\frac{s}{t}\Big)^{2\rho} ds \;\le\; \int_0^t e^{2\omega(t-s)}\, ds \;=\; e^{2\omega t}\cdot \frac{e^{-2\omega t}-1}{2|\omega|} \;\le\; \frac{1}{2|\omega|} \qquad\qquad (\text{for } \rho \ge 0)$$

For $\rho < 0$, still with $\omega < 0$,

$$\int_a^t e^{2\omega(t-s)} \Big(\frac{s}{t}\Big)^{2\rho} ds \;\le\; \int_{t/2}^t e^{2\omega(t-s)} \Big(\frac{t/2}{t}\Big)^{2\rho} ds + \int_0^{t/2} e^{2\omega(t-s)} \Big(\frac{1}{t}\Big)^{2\rho} ds \qquad\qquad (\text{for } \rho < 0)$$

The two integrals are bounded in $t \ge a$, for elementary reasons. Thus, for $\mathrm{Re}(\omega) < 0$, the kernel $K(x,t)$ is bounded.

[4.3] $\mathrm{Re}(\omega) = 0$ and cancellation

When $\mathrm{Re}(\omega) = 0$, absolute value estimates no longer suffice to prove boundedness. Cancellation must be exploited by an integration by parts. Choose $\pm\omega$ so that $\mathrm{Re}(\rho) \ge 0$. One integration by parts gives

$$\int_x^t e^{2\omega(t-s)} \Big(\frac{s}{t}\Big)^{2\rho} ds \;=\; \Big[\frac{e^{2\omega(t-s)}}{-2\omega} \Big(\frac{s}{t}\Big)^{2\rho}\Big]_x^t + \int_x^t \frac{e^{2\omega(t-s)}}{2\omega}\, \frac{2\rho}{s} \Big(\frac{s}{t}\Big)^{2\rho} ds$$

$$=\; \frac{1}{-2\omega} - \frac{e^{2\omega(t-x)}}{-2\omega} \Big(\frac{x}{t}\Big)^{2\rho} + \int_x^t \frac{e^{2\omega(t-s)}}{2\omega}\, \frac{2\rho}{s} \Big(\frac{s}{t}\Big)^{2\rho} ds$$

The leading terms are bounded for $t \ge x \ge a$. The latter integral can be estimated by absolute values, for $\mathrm{Re}\rho \ne 0$:

$$\Big|\int_x^t e^{2\omega(t-s)} \frac{1}{s}\Big(\frac{s}{t}\Big)^{2\rho} ds\Big| \;\le\; \int_x^t \frac{1}{s}\Big(\frac{s}{t}\Big)^{2\,\mathrm{Re}\,\rho} ds \;=\; \frac{1}{2\,\mathrm{Re}\rho}\Big(1 - \Big(\frac{x}{t}\Big)^{2\,\mathrm{Re}\,\rho}\Big)$$

When $\mathrm{Re}\rho = 0$, a second integration by parts gives

$$\int_x^t e^{2\omega(t-s)}\, \frac{1}{s}\Big(\frac{s}{t}\Big)^{2\rho} ds \;=\; \Big[\frac{e^{2\omega(t-s)}}{-2\omega}\, \frac{1}{s}\Big(\frac{s}{t}\Big)^{2\rho}\Big]_x^t + \frac{2\rho-1}{2\omega} \int_x^t e^{2\omega(t-s)}\, \frac{1}{s^2}\Big(\frac{s}{t}\Big)^{2\rho} ds$$

The latter integral is estimated by

$$\Big|\int_x^t e^{2\omega(t-s)}\, \frac{1}{s^2}\Big(\frac{s}{t}\Big)^{2\rho} ds\Big| \;\le\; \int_x^t \frac{ds}{s^2} \;=\; \frac{1}{x} - \frac{1}{t} \;\le\; \frac{1}{a}$$

Thus, in all cases, $K(x,t)$ is bounded on $t \ge x \ge a > 0$.

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5. *End of construction of solutions*

[5.1] Bound for T

As observed above, there is a bound A so that $|F(x)| \le A$ for $x \ge a$. Let $|K(x,t)| \le B$. For f satisfying a bound $|f(x)| \le x^{-\lambda}$ for $x \ge a$, with $\lambda > -1$,

$$|(Tf)(x)| \;\le\; \frac{AB}{\lambda+1} x^{-(\lambda+1)} \qquad\qquad (\text{for } x \ge a)$$

Indeed,

$$|Tf(x)| \;=\; \Big|\int_x^\infty K(x,t)\,F(t)\,f(t)\,\frac{dt}{t^2} \;\le\; AB\int_x^\infty t^{-(\lambda+2)}\,dt$$

[5.2] Bound on f_n

With $f_0 = 1$ and $f_{n+1} = Tf_n$, we claim that

$$|f_n(x)| \;\le\; \frac{(AB)^n}{n!}x^{-n} \qquad\qquad (\text{for } n = 0,1,2,\ldots \text{ and } x \ge a)$$

This holds for $n = 0$, and induction using the bound on T gives the result.

[5.3] Convergence of the series

Now we show that the series

$$f(x) \;=\; \sum_{n\ge 0} f_n(x) \;=\; \sum_{n\ge 0} T^n f_0(x)$$

converges uniformly absolutely, and satisfies the integral equation

$$f(x) \;=\; 1 + \int_x^\infty K(x,t)\,F(t)\,f(t)\,\frac{dt}{t^2}$$

Uniform absolute convergence in $C^o[a,+\infty)$ follows from the previous estimate. This justifies interchange of summation and integration:

$$Tf(x) \;=\; \int_x^\infty K(x,t)\,F(t)\,f(t)\,\frac{dt}{t^2} \;=\; \sum_{n\ge 0}\int_x^\infty K(x,t)\,F(t)\,T^n f_0(t)\,\frac{dt}{t^2}$$

$$=\; \sum_{n\ge 0} T^{n+1} f_0(x) \;=\; -1 + \sum_{n\ge 0} T^n f_0(x) \;=\; -1 + f(x)$$

Thus, f satisfies the integral equation. Since $K(x,t)$ is differentiable in x, and since the integral for T converges well, the expression

$$f(x) \;=\; 1 + \int_x^\infty K(x,t)\,F(t)\,f(t)\,\frac{dt}{t^2}$$

demonstrates the differentiability of f. Further, since $K(x,x) = 0$, the derivative is

$$f'(x) \;=\; \int_x^\infty \frac{\partial K(x,t)}{\partial x}\,F(t)\,f(t)\,\frac{dt}{t^2} \;=\; \int_x^\infty e^{2\omega(t-x)}\Big(\frac{x}{t}\Big)^{2\rho} F(t)\,f(t)\,\frac{dt}{t^2}$$

The integral is again continuously differentiable in x, so f is in C^2.

[5.4] Back to the differential equation

From the integral expression,

$$f''(x) \;=\; -\frac{F(x)}{x^2}\,f(x) \;+\; \int_x^\infty \Big(-2\omega + \frac{2\rho}{x}\Big)e^{2\omega(t-x)}\Big(\frac{x}{t}\Big)^{2\rho} F(t)\,f(t)\,\frac{dt}{t^2}$$

Substituting into the differential equation,

$$f'' + 2\big(\omega - \frac{\rho}{x}\big)f' \;+\; \frac{F}{x^2}\,f \;=$$

$$-\frac{F}{x^2}\,f \;+\; \int_x^\infty \Big(-2\omega + \frac{2\rho}{x}\Big) e^{2\omega(t-x)} \left(\frac{x}{t}\right)^{2\rho} F(t)\,f(t)\,\frac{dt}{t^2}$$

$$+\,2\Big(\omega - \frac{\rho}{x}\Big)\int_x^\infty e^{2\omega(t-x)} \left(\frac{x}{t}\right)^{2\rho} F(t)\,f(t)\,\frac{dt}{t^2} \;+\; \frac{F}{x^2}\,f \;=\; 0$$

Then

$$u(x) \;=\; e^{\omega x} x^{-\rho}\, f(x)$$

satisfies the original equation

$$u'' - q(x)\,u \;=\; 0$$

[5.5] Two independent solutions

In the special case that $q_o < 0$ and $q_1 \in \mathbb{R}$, $\omega = \sqrt{\omega}$ has $\mathrm{Re}\,\omega = 0$ and $\mathrm{Re}\,\rho = 0$. In that case, the successive approximation solution to the integral equation can proceed with either values $\pm\omega$, $\pm\rho$, and two linearly independent solutions are obtained.

In all other cases, the successive approximation argument succeeds for only one choice of sign, producing a solution u as above. Nevertheless, a second solution can be constructed as follows, by a standard device. Since $f(x) = 1 + O(1/x)$, there is $b \geq a$ large enough so that $u(x) \neq 0$ for $x \geq b$. Then let $v = u \cdot w$, require that v satisfy $v'' - q\,v = 0$, and see what condition this imposes on w. From

$$v'' - q\,v \;=\; u''w + 2u'w' + uw'' - q\,uw \;=\; 0$$

using $u'' - q\,u = 0$, we obtain

$$\frac{w''}{w'} \;=\; \frac{-2u'}{u}$$

Then

$$\log w' \;=\; -2\log u + C$$

and

$$w(x) \;=\; \int_b^x u(t)^{-2}\,dt$$

Thus, a second solution is

$$u(x)\cdot\int_b^x u(t)^{-2}\,dt$$

That integral is not constant, so the two solutions are linearly independent.

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6. *Asymptotics of solutions*

We show that the solutions on $x \geq a$ have the same asymptotics as the heuristic indicated earlier.

[6.1] Some elementary asymptotics

Use the standard device $(\rho)_\ell = \rho(\rho{+}1)\ldots(\rho{+}\ell{-}1)$ and $(\rho)_0 = 1$. Let $0 \neq \omega \in \mathbb{C}$ with $\mathrm{Re}\,\omega \leq 0$. If $\mathrm{Re}\,\omega = 0$, require that $\mathrm{Re}\,\rho > 1$. Repeated integration by parts and easy estimates yield asymptotic expansions,

$$\begin{cases} \displaystyle\int_x^\infty e^{\omega t}\, t^{-\rho}\,dt & \sim \quad e^{\omega x}\cdot \displaystyle\sum_{\ell\geq 0} \frac{(\rho)_\ell}{(-\omega)^{\ell+1}}\,\frac{1}{x^{\rho+\ell}} \\ \displaystyle\int_b^x e^{-\omega t}\, t^{-\rho}\,dt & \sim \quad e^{-\omega x}\cdot \displaystyle\sum_{\ell\geq 0} \frac{(\rho)_\ell}{\omega^{\ell+1}}\,\frac{1}{x^{\rho+\ell}} \end{cases}$$

Since the sup of $|e^{\omega t} t^{-\rho}|$ occurs farther to the right for larger $\mathrm{Re}(\rho) < 0$, these asymptotics are *not uniform* in ρ. Note that the boundedness of the kernel $K(x,t)$ proven earlier has a weaker hypothesis than the second asymptotic assertion, requires a slightly more complicated argument, and has a weaker conclusion.

[6.2] Asymptotics of $T^n f_0$

With $f_0 = 1$, we claim that $f_n = T^n f_0$ has an asymptotic expansion at $+\infty$, of the form

$$f_n \sim \sum_{\ell \geq n} c_{n\ell}\, x^{-\ell}$$

This holds for $f_0 = 1$. To do the induction step, assume f_n has such an asymptotic expansion. Then $F(x) \cdot f_n(x)$ has a similar expansion

$$F\, f_n \sim \sum_{\ell \geq n} b_\ell\, x^{-\ell}$$

because [8]

$$F(x) \;=\; x^2 \cdot \Big(\omega^2 - \frac{2\omega\rho}{x} + \frac{\rho^2+\rho}{x^2} - q(x)\Big)$$

and q is assumed to have an asymptotic expansion in the functions $1/x^n$ at $+\infty$. We want to insert the asymptotic expansion for $F\, f_n$ into the integral in the differentiated form of $f_{n+1} = T f_n$, namely, into the equation

$$f'_{n+1}(x) \;=\; \int_x^\infty e^{2\omega(t-x)} \Big(\frac{x}{t}\Big)^{2\rho} F(t)\, f_n(t)\, \frac{dt}{t^2}$$

Indeed, from

$$F(x)\, f_n(x) - \sum_{n \leq \ell \leq N} b_\ell\, x^{-\ell} \;=\; O(x^{-(N+1)})$$

and from the boundedness of $K(x,t)$ we have

$$\Big| \int_x^\infty e^{2\omega(t-x)} \Big(\frac{x}{t}\Big)^{2\rho} \Big(F(t)\, f_n(t) - \sum_{n \leq \ell \leq N} b_\ell\, t^{-\ell}\Big) \frac{dt}{t^2} \Big| \;=\; \Big| \int_x^\infty e^{2\omega(t-x)} \Big(\frac{x}{t}\Big)^{2\rho} O(t^{-(N+1)})\, \frac{dt}{t^2} \Big|$$

$$\ll_{\omega,\rho,N}\; x^{-(N+1)} \int_x^\infty \frac{dt}{t^2} \;=\; O(x^{-(N+2)}) \;=\; o(x^{-(N+1)})$$

Thus, the desired asymptotics for f'_{n+1} would follow from asymptotics for the collection

$$\int_x^\infty e^{2\omega(t-x)} \Big(\frac{x}{t}\Big)^{2\rho} \Big(\sum_{n \leq \ell \leq N} b_\ell\, t^{-\ell}\Big) \frac{dt}{t^2} \qquad\qquad (\text{for } N \geq n)$$

As noted above,

$$\int_x^\infty e^{\omega t}\, t^{-\rho}\, dt \;\sim\; e^{\omega x} \cdot \sum_{\ell \geq 0} \frac{(\rho)_\ell}{(-\omega)^{\ell+1}}\, \frac{1}{x^{\rho+\ell}}$$

Note that for each N only finitely-many asymptotic expansions are used, so *uniformity* is not an issue. After some preliminary rearrangements, this gives

$$\int_x^\infty e^{2\omega(t-x)} \Big(\frac{x}{t}\Big)^{2\rho} \Big(\sum_{n \leq \ell \leq N} b_\ell\, t^{-\ell}\Big) \frac{dt}{t^2} \;=\; \sum_{n \leq \ell \leq N} b_\ell \int_x^\infty e^{2\omega(t-x)} \Big(\frac{x}{t}\Big)^{2\rho} t^{-\ell}\, \frac{dt}{t^2}$$

[8] The product of two asymptotic expansions in $1/x^n$ is readily shown to be an asymptotic expansion for the product function.

$$= \sum_{n\le \ell\le N} b_\ell \, e^{-2\omega x} x^{2\rho} \int_x^\infty e^{2\omega t}\, t^{-(2\rho+\ell+2)}\, dt$$

$$= \sum_{n\le \ell\le N} b_\ell \, e^{-2\omega x} x^{2\rho} \cdot e^{2\omega x} \Big(\sum_{0\le m\le N-(2+\ell)} \frac{(\rho+\ell+2)_m}{(-2\omega)^{m+1}} \, \frac{1}{x^{2\rho+\ell+2+m}} \;+\; O\Big(\frac{1}{x^{2\rho+N+1}}\Big)\Big)$$

$$= \sum_{n\le \ell\le N} b_\ell \sum_{0\le m\le N-(2+\ell)} \frac{(\rho+\ell+2)_m}{(-2\omega)^{m+1}} \, \frac{1}{x^{\ell+2+m}} \;+\; O\Big(\frac{1}{x^{N+1}}\Big)$$

This holds for all N, so we have an asymptotic expansion for f'_{n+1}:

$$f'_{n+1}(x) \;\sim\; \sum_{k\ge n+1} \Big(\sum_{\ell\,:\, n\le \ell\le k} b_\ell \frac{(\rho+\ell+2)_{k-\ell}}{(-2\omega)^{m+1}} \Big) \frac{1}{x^{k+2}}$$

Integrating this in x gives the asymptotic expansion of f_{n+1}. (See the appendix.)

[6.3] Asymptotics of the solution f

Obviously we expect the asymptotic expansion of $f = \sum_n f_n$ to be the sum of those of f_n, all the more so since the $1/x^m$ terms in the expansion of f_n vanish for $m < n$. The uniform pointwise bound

$$|f_n(x)| \;\le\; \frac{(AB)^n}{n!} x^{-n} \qquad\qquad \text{(for } n = 0, 1, 2, \ldots \text{ and } x \ge a)$$

proven earlier legitimizes this. Thus, the solution f has an asymptotic expansion of the desired type.

To prove that this asymptotic expansion is the same as the expansion obtained by a recursion earlier, we show that the coefficients satisfy the same recursion.

The integral expression for f' in terms of f (above) proves that f' has an asymptotic expansion, and similarly for f''. As proven in the appendix, this justifies two termwise differentiations of the asymptotic for f.

The asymptotics for f, f', and f'' can be inserted in the differential equation

$$f'' + 2\big(\omega - \frac{\rho}{x}\big) f' + \Big(\omega^2 - \frac{2\omega\rho}{x} + \frac{\rho^2+\rho}{x^2} - q(x)\Big) f \;=\; 0$$

for f. We have assumed that the coefficient of f has an asymptotic expansion, and this equation gives the expected recursive relation on the coefficients of the asymptotic for f. Therefore, the solution

$$u(x) \;=\; e^{\omega x}\, x^{-\rho}\, f(x)$$

to the original differential equation has the asymptotics inherited from f, which match the heuristic asymptotics from the earlier formal/heuristic solution.

[6.4] The second solution

Now we show that the second solution

$$v(x) \;=\; u(x) \cdot \int_b^x u(t)^{-2}\, dt$$

to the original differential equation has the asymptotics given by the heuristic recursion, but with the opposite choice of $\pm\omega$ and $\pm\rho$. In terms of f,

$$v(x) \;=\; u(x) \cdot \int_b^x u(t)^{-2}\, dt \;=\; e^{\omega x}\, x^{-\rho}\, f(x) \int_b^x e^{-2\omega t}\, x^{-2\rho}\, f(t)\, dt$$

$$= e^{-\omega x}\, x^\rho\, f(x) \int_b^x e^{2\omega(x-s)} \Big(\frac{s}{x}\Big)^{2\rho} f(s)^{-2}\, ds$$

This motivates taking

$$g(x) \;=\; f(x) \int_b^x e^{2\omega(x-s)} \Big(\frac{s}{x}\Big)^{2\rho} f(s)^{-2}\, ds$$

The lower bound b has been chosen large enough so that $f(x)$ is bounded away from 0 for $x \geq b$. Since f has an asymptotic expansion with leading coefficient 1, it is elementary that there are coefficients a_n so that $1/f^2$ has asymptotics

$$\frac{1}{f(x)^2} \;=\; 1 + \sum_{1\leq n\leq N} \frac{a_n}{x^n} + O\big(\frac{1}{x^{N+1}}\big) \qquad\qquad (\text{with } a_0 = 1)$$

Then

$$\frac{g(x)}{f(x)} \;=\; \sum_{0\leq n\leq N} a_n \int_b^x e^{2\omega(x-s)} \Big(\frac{s}{x}\Big)^{2\rho} \frac{1}{s^n}\, ds + \int_b^x e^{2\omega(x-s)} \Big(\frac{s}{x}\Big)^{-2\rho} O\big(\frac{1}{s^{N+1}}\big)\, ds$$

The last integral is $O(1/x^{N+1})$, from the elementary asymptotics. For each fixed N, the finitely-many integrals inside the summation have elementary asymptotics. Since for fixed N there are only finitely-many such asymptotics, they are trivially *uniform*, so the asymptotics can be added. The asymptotic expansion for

$$\int_b^x e^{2\omega(x-s)} \Big(\frac{s}{x}\Big)^{2\rho} \frac{1}{s^n}\, ds$$

begins with $1/x^n$, so the coefficient of each $1/x^n$ is a finite sum, and there is no issue of convergence. Multiplying this asymptotic by that of $f(g)$ give the asymptotic expansion of $g(x)$.

As with f, the derivatives g' and g'' of g have integral representations which yield asymptotic expansions. Thus, as in the appendix, the asymptotic expansion for g can be twice differentiated term-wise to give those of g' and g''. Thus, their asymptotic expansions can be inserted in the differential equation. Their coefficients must satisfy the same recursion with some choice of $\pm\omega$ and corresponding $\pm\rho$. Arguing that the asymptotic for g cannot be identical to that of f, we infer that the recursion for the coefficients of g uses the opposite choice $-\omega, -\rho$ from the choice ω, ρ used to construct f.

[6.4.1] Remark: When ω and ρ are both purely imaginary, u and v are bounded, neither approaches 0, and they are uniquely determined up to constant factors. In all other cases, one solution approaches 0, and is uniquely determined up to a constant, while the other is unbounded and ambiguous by multiples of the first, insofar as it depends on the choice of lower bound b in the integral above.

[6.4.2] Remark: Stokes' phenomenon When the coefficient $q(x)$ of the differential equation $u'' - q(x)u = 0$ is *analytic* in a sector in $\mathbb{C}$, and when q admits the same sort of asymptotic expansion

$$q(e^{i\theta}x) \;\sim\; \sum_{n\geq 0} \frac{q_n\, e^{-in\theta}}{x^n} \qquad\qquad (\text{uniformly in } \theta)$$

in that sector, *uniformly in the argument* θ, with $q_o \neq 0$, the above discussion still applies. In the real-variable discussion, with $\omega = \pm\sqrt{q_o}$, the case $\mathrm{Re}\,\omega = 0$ was at the interface between the regimes $\mathrm{Re}\,\omega \leq 0$ and $\mathrm{Re}\,\omega \geq 0$ in which behaviors of solutions differed. Similarly, in the complex-variable situation the line $\mathrm{Re}\,(z \cdot \sqrt{q_o}) = 0$ is the boundary between regimes of different behavior. *On* that line, the behavior is as in the $\mathrm{Re}\,\omega = 0$ case. On either side of that line, one solution is exponentially larger than the other, etc. This is *Stokes' phenomenon.*

7. Appendix: asymptotic expansions

We essentially follow [Erdélyi 1956].

To say that φ_ℓ is an *asymptotic sequence* at x_o means that $\varphi_{\ell+1}(x) = o(\varphi_\ell(x))$ as $x \to x_o$, for all ℓ. A function f has an *asymptotic expansion* in terms of the φ_n, expressed with coefficients c_n as

$$f(x) \;\sim\; \sum_{n\geq 0} c_n\varphi_n$$

when, for all $N \geq 0$,

$$f(x) - \sum_{0\leq n\leq N} c_n\varphi_n \;=\; o(\varphi_N)$$

It is not surprising that a sum or integral of asymptotic expansions *uniform* in a parameter has the expected asymptotics. Circumstances under which an asymptotic expansion can be *differentiated* are more special.

[7.1] Summing asymptotic expansions

Let functions f_n have asymptotic expansions $f_n \sim \sum_{\ell\geq 0} c_{n\ell}\,\varphi_\ell$, *uniform* in n, meaning that

$$f_n(x) - \sum_{\ell\leq N} c_{n\ell}\,\varphi_\ell \;=\; o(\varphi_N) \qquad \text{(implied constant and neighborhood of } x_o \textit{ uniform} \text{ in } n)$$

Let a_n be coefficients such that $\sum_n a_n \cdot c_{n\ell}$ is convergent and $\sum_n a_n$ is *absolutely* convergent. We claim that $\sum_n a_n f_n$ converges in a neighborhood of x_o and has the expected asymptotic expansion

$$\sum_n a_n\, f_n \;\sim\; \sum_\ell \Big(\sum_n a_n\, c_{n\ell}\Big)\,\varphi_n$$

The *uniformity* of the asymptotic expansions, and $\sum_n |a_n| < \infty$, give

$$\sum_{n\geq 1} a_n\,\Big(f_n(x) - c_{n1}\,\varphi_1(x)\Big) \;=\; o(\varphi_1(x)) \qquad \text{(uniformly in } x)$$

In particular, the sum on the left-hand side converges for fixed x. Since $\sum_n a_n c_{n1}$ converges, $\sum_{n\geq 1} a_n\, f_n(x)$ converges. Similarly,

$$\sum a_n f_n(x) - \sum_{\ell\leq N} \big(\sum_n a_n c_{n\ell}\big)\,\varphi_\ell \;=\; o(\varphi_N)$$

[7.2] Integrals

The general case is readily extrapolated from the example of an infinite sum. Namely, let $f(x,y) \sim \sum_\ell c_\ell(y)\,\varphi_\ell$ be asymptotic expansions *uniform* in a parameter $y \in Y$, where Y is a measure space. Suppose that $y \to f(x,y)$ is measurable for each x, and that every $c_\ell(y)$ is measurable. Let $a(y)$ be absolutely integrable on Y, and assume that the integrals

$$\int_Y a(y)\,c_\ell(y)\,dy$$

converge for all n. Then

$$\int_Y a(y)\,f(x,y)\,dy$$

exists for x close to x_o, and has asymptotic expansion

$$\int_Y a(y)\, f(x,y)\, dy \;\sim\; \sum_\ell \Big(\int_Y a(y)\, c_\ell(y)\, dy \Big) \varphi_\ell$$

[7.3] Differentiation of asymptotics in $1/x^n$

Asymptotic power series are asymptotic expansions

$$f(x) \;\sim\; c_o + \frac{c_1}{x} + \frac{c_2}{x^2} + \ldots \qquad\qquad (\text{as } x \to +\infty)$$

Unlike general situations, two such asymptotic expansions can be *multiplied*. A special property of asymptotic power series is the *absolute integrability* of $f(x) - c_o - c_1/x = O(x^{-2})$ on intervals $[a, +\infty)$. Let

$$F(x) \;=\; \int_x^\infty \big(f(t) - c_o - \frac{c_1}{t}\big)\, dt$$

We claim that F has an asymptotic expansion obtained from that of $f(x) - c_o - c_1/x$ by integrating termwise, namely,

$$F(x) \;\sim\; \frac{c_2}{t} + \frac{c_3}{2t^2} + \frac{c_4}{3t^3} + \ldots$$

To prove this, use

$$f(x) - \Big(c_o + \frac{c_1}{x} + \ldots + \frac{c_N}{x^N}\Big) \;=\; O(x^{-(N+1)})$$

Then

$$F(x) - \Big(\frac{c_2}{t} + \frac{c_3}{2t^2} + \ldots + \frac{c_N}{Nx^{N-1}}\Big) \;=\; \int_x^\infty \big(f(t) - c_o - \frac{c_1}{t}\big)\, dt - \int_x^\infty \big(\frac{c_2}{t^2} + \frac{c_3}{t^3} + \ldots + \frac{c_N}{x^N}\big)\, dt$$

$$= \int_x^\infty O(t^{-(N+1)})\, dt \;=\; O(x^{-N}) \;=\; o(x^{-(N-1)})$$

This has a surprising corollary about differentiation: for f with an asymptotic power series at $+\infty$ as above, if f is differentiable, and if f' has an asymptotic power series at $+\infty$, then the asymptotics of f' are obtained by differentiating that of f termwise:

$$f'(x) \;\sim\; -\frac{c_1}{x^2} - \frac{2c_2}{x^3} - \frac{3c_3}{x^4} - \ldots$$

When f is *holomorphic* in a region in which the asymptotic holds *uniformly* in the argument of x, Cauchy's integral formula for f' produces an asymptotic for f' from that for f, thus avoiding the need to make a hypothesis that f' admits an asymptotic expansion.

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