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Asymptotics of integrals

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The simplest notion of *asymptotic* $F(s)$ for $f(s)$ as s goes to $+\infty$ on $\mathbb{R}$, or in a sector in $\mathbb{C}$, is a simpler function $F(s)$ such that $\lim_s f(s)/F(s) = 1$, written $f \sim F$. One might want an error estimate, for example,

$$f \sim F \iff f(s) = F(s) \cdot (1 + O(\frac{1}{|s|}))$$

That is,

$$f(s) \;\sim\; f_0(s) \cdot \big(\frac{c_0}{s^\alpha} + \frac{c_1}{s^{\alpha+1}} + \frac{c_2}{s^{\alpha+2}} + \ldots\big)$$

(with an auxiliary function f_0) is an *asymptotic expansion* for f when

$$f \;=\; f_0(s) \cdot \big(\frac{c_0}{s^\alpha} + \frac{c_1}{s^{\alpha+1}} + \ldots + \frac{c_n}{s^{\alpha+n}} + O\big(\frac{1}{|s|^{\alpha+n+1}}\big)\big)$$

We consider two of the simplest methods to obtain asymptotics of integrals: *Watson's lemma* and *Laplace's method*. Watson's lemma dates from at latest [Watson 1918a], and Laplace's method at latest from [Laplace 1774].

An important example is the Stirling-Laplace asymptotic for $\Gamma(s)$:

$$\Gamma(s) \;\sim\; \sqrt{2\pi}\, e^{-s}\, s^{s-\frac{1}{2}} \qquad\qquad (\text{as } |s| \to \infty, \text{ with } \mathrm{Re}(s) \ge \delta > 0)$$

A useful result about ratios of gamma functions:

$$\frac{\Gamma(s+a)}{\Gamma(s)} \;\sim\; s^a \qquad\qquad (\text{as } |s| \to \infty, \text{ for fixed } a, \text{ for } \mathrm{Re}(s) \ge \delta > 0)$$

The latter is very awkward to obtain as a corollary from Stirling's formula. Laplace's method is further illustrated by functions closely related to Bessel functions, namely, for any fixed spectral parameter $\nu \in \mathbb{R}$,

$$\sqrt{y} \int_0^\infty e^{-(u+\frac{1}{u})y}\, u^{i\nu}\, \frac{du}{u} \;\sim\; \sqrt{\pi} \cdot e^{-2y} \qquad\qquad (\text{as } y \to +\infty)$$

To the extent possible, we want to understand the asymptotics of gamma and other important special functions on general principles.

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1. *Heuristic for Stirling's asymptotic*

First we give a heuristic for the main term of the Laplace-Stirling asymptotic, namely

$$\Gamma(s) \;\sim\; e^{-s} \cdot s^{s-\frac{1}{2}} \cdot \sqrt{2\pi}$$

Using Euler's integral,

$$s \cdot \Gamma(s) \;=\; \Gamma(s+1) \;=\; \int_0^\infty e^{-u}\, u^{s+1}\, \frac{du}{u} \;=\; \int_0^\infty e^{-u}\, u^s\, du \;=\; \int_0^\infty e^{-u+s\log u}\, du$$

The trick is to replace the exponent $-u + s\log u$ by the quadratic polynomial in u best approximating it near its maximum, and evaluate the resulting integral. This replacement is justified via Watson's lemma and Laplace's method, below, but the heuristic is simpler than the justification.

The exponent takes its maximum where its derivative vanishes, at the unique solution $u_o = s$ of

$$-1 + \frac{s}{u} \;=\; 0$$

The second derivative in u of the exponent is $-s/u^2$, which takes value $-1/s$ at $u_o = s$. Thus, near $u_o = s$, the quadratic Taylor-Maclaurin polynomial in t approximating the exponent is

$$-s + s\log s - \frac{1}{2!\, s} \cdot (u-s)^2$$

We imagine that

$$s \cdot \Gamma(s) \;\sim\; \int_0^\infty e^{-s+s\log s - \frac{1}{2s}\cdot(u-s)^2}\, du \;=\; e^{-s} \cdot s^s \cdot \int_{-\infty}^\infty e^{-\frac{1}{2s}\cdot(u-s)^2}\, du$$

The latter integral is taken over the whole real line. Evaluation of the integral over the whole line, and simple estimates on the integral over $(-\infty, 0]$, show that the integral over $(-\infty, 0]$ is of a lower order of magnitude than the whole. Thus, the leading term of the asymptotics of the integral over the whole line is the same than the integral from 0 to $+\infty$. To simplify the remaining integral, replace u by su and cancel a factor of s from both sides,

$$\Gamma(s) \;\sim\; e^{-s} \cdot s^s \cdot \int_{-\infty}^\infty e^{-s(u-1)^2/2}\, du$$

Replace u by $u+1$, and u by $u \cdot \sqrt{2\pi/s}$, obtaining

$$\int_{-\infty}^\infty e^{-s(u-1)^2/2}\, du \;=\; \int_{-\infty}^\infty e^{-su^2/2}\, du \;=\; \frac{\sqrt{2\pi}}{\sqrt{s}} \int_{-\infty}^\infty e^{-\pi u^2}\, du \;=\; \frac{\sqrt{2\pi}}{\sqrt{s}}$$

and

$$\Gamma(s) \;\sim\; e^{-s} \cdot s^{s-\frac{1}{2}} \cdot \sqrt{2\pi}$$

This heuristic can be made rigorous, as below.

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2. *Watson's lemma*

The often-rediscovered *Watson's lemma* [1] gives an asymptotic expansion for certain Laplace transforms, valid in half-planes in $\mathbb{C}$. For example, let h be a smooth function on $(0, +\infty)$ all whose derivatives are of polynomial growth, and expressible for small $x > 0$ as

$$h(x) \;=\; x^\alpha \cdot g(x)$$

[1] This lemma appeared in the treatise [Watson 1922] on page 236, citing [Watson 1918a], page 133. Curiously, the aggregate bibliography of [Watson 1922] omitted [Watson 1918a], and the footnote mentioning it gave no title. Happily, [Watson 1918a] is mentioned by title in [Blaustein-Handelsman 1975]. We mention [Watson 1917], [Watson 1918a], [Watson 1918b], for perspective.

for some $\alpha \in \mathbb{C}$, where $g(x)$ is differentiable on $\mathbb{R}$ near 0. We do *not* need to assume that g is *real-analytic* near 0, only that it and its derivatives have finite Taylor expansions approximating it well as $x \to 0^+$. Thus, $h(x)$ has an expression

$$h(x) \;=\; x^\alpha \cdot \sum_{n=0}^\infty c_n\, x^n \qquad\qquad \text{(for } 0 < x \text{ sufficiently small)}$$

Then there is an *asymptotic expansion* of the Laplace transform of h,

$$\int_0^\infty e^{-xs}\, h(x)\, \frac{dx}{x} \;\sim\; \frac{\Gamma(\alpha)\, c_0}{s^\alpha} + \frac{\Gamma(\alpha+1)\, c_1}{s^{\alpha+1}} + \frac{\Gamma(\alpha+2)\, c_2}{s^{\alpha+2}} + \ldots \qquad\qquad \text{(for } \mathrm{Re}\,(s) > 0)$$

A simple corollary of the error estimates given below is that, letting $\mathrm{Re}\,(\alpha) + 1 - \varepsilon$ be the greatest integer less than or equal $\mathrm{Re}\,(\alpha) + 1$,

$$\int_0^\infty e^{-xs}\, h(x)\, \frac{dx}{x} \;=\; \int_0^\infty e^{-xs}\, x^\alpha\, g(x)\, \frac{dx}{x} \;=\; \frac{\Gamma(\alpha)\, g(0)}{s^\alpha} + O\Big(\frac{1}{|s|^{\mathrm{Re}\,(\alpha)+1-\varepsilon}}\Big)$$

Since

$$\mathrm{Re}\,(\alpha) + 1 - \varepsilon \;>\; \mathrm{Re}\,(\alpha)$$

the error term is of strictly smaller order of magnitude in s.

The idea of the proof is straightforward: the expansion is obtained from

$$\int_0^\infty e^{-xs}\, h(x)\, \frac{dx}{x} \;=\; \int_0^\infty e^{-xs}\, x^\alpha \big(c_0 + \ldots + c_n x^n\big)\, \frac{dx}{x} \;+\; \int_0^\infty e^{-xs}\, x^\alpha \Big(g(x) \;-\; \big(c_0 + \ldots + c_n x^n\big)\Big)\, \frac{dx}{x}$$

The first integral gives the asymptotic expansion, and for $\mathrm{Re}\,(s) > 0$ the second integral can be integrated by parts essentially $\mathrm{Re}\,(\alpha) + n$ times and trivially bounded to give a $O(1/s^{\alpha+n-\varepsilon})$ error term for some small $\varepsilon \geq 0$. For the integration by parts the denominator x in the measure must be moved into the integrand proper, accounting for a slight reduction of the order of vanishing of the integrand at 0.

To understand the error, let $\varepsilon \geq 0$ be the smallest such that

$$N \;=\; \mathrm{Re}\,(\alpha) + n - \varepsilon \;\in\; \mathbb{Z}$$

The subtraction of the initial polynomial and re-allocation of the $1/x$ from the measure makes $x^{\alpha-1}(g(x) - (c_0 + \ldots + c_n x^n))$ vanish to order N at 0. This, with the exponential e^{-sx} and the presumed polynomial growth of h and its derivatives, allows integration by parts N times without boundary terms, giving

$$\int_0^\infty e^{-xs}\, h(x)\, dx \;=\; \frac{\Gamma(\alpha)\, c_0}{s^\alpha} + \frac{\Gamma(\alpha+1)\, c_1}{s^{\alpha+1}} + \ldots + \frac{\Gamma(\alpha+n)\, c_n}{s^{\alpha+n}}$$
$$+\frac{1}{s^N} \int_0^\infty e^{-sx} \Big(\frac{\partial}{\partial x}\Big)^N \Big(x^\alpha \cdot \big(g(x) - (c_0 + \ldots + c_n x^n)\big)\Big)\, dx$$

The last error-like term is $O(s^{-[\mathrm{Re}\,(\alpha)+n-\varepsilon]})$. That is, computing in this fashion, the error term swallows up the last term in the asymptotic expansion.

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3. *Watson's lemma illustrated on $B(s,a)$*

Here is an asymptotic result non-trivial to derive from Stirling's formula for $\Gamma(s)$, but easy to obtain from Watson's lemma. Euler's beta integral is

$$B(s,a) \;=\; \int_0^1 x^{s-1}\, (1-x)^{a-1}\, dx \;=\; \frac{\Gamma(s)\, \Gamma(a)}{\Gamma(s+a)}$$

Fix a with $\mathrm{Re}(a) > 0$, and consider this integral as a function of s. Letting $x = e^{-u}$ gives an integrand fitting Watson's lemma,

$$B(s,a) \;=\; \int_0^\infty e^{-su}\,(1-e^{-u})^{a-1}\,du \;=\; \int_0^\infty e^{-su}\,(u - \frac{u^2}{2!} + \ldots)^{a-1}\,du$$

$$= \int_0^\infty e^{-su}\,u^a \cdot (1 - \frac{u}{2!} + \ldots)^{a-1}\,\frac{du}{u} \;\sim\; \frac{\Gamma(a)}{s^a}$$

taking just the first term in an asymptotic expansion, using Watson's lemma. Thus,

$$\frac{\Gamma(s)\,\Gamma(a)}{\Gamma(s+a)} \;\sim\; \frac{\Gamma(a)}{s^a}$$

giving

$$\frac{\Gamma(s)}{\Gamma(s+a)} \;\sim\; \frac{1}{s^a} \qquad\qquad (\text{for } a \text{ fixed})$$

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4. *Simple form of Laplace's method, and* $\Gamma(s)$

Laplace's method obtains asymptotics in s for certain integrals of the form

$$\int_0^\infty e^{-s\cdot f(u)}\,du$$

with f real-valued. The idea is that the *minimum values* of $f(u)$ should dominate, and the leading term of the asymptotics should be

$$\int_0^\infty e^{-s\cdot f(u)}\,du \;\sim\; e^{-sf(u_o)} \cdot \frac{\sqrt{2\pi}}{\sqrt{f''(u_o)}} \cdot \frac{1}{\sqrt{s}} \qquad\qquad (\text{for } |s| \to \infty, \text{ with } \mathrm{Re}(s) \ge \delta > 0)$$

To reduce this to Watson's lemma, break the integral at points where the derivative f' changes sign, and change variables to convert each fragment to a Watson-lemma integral. For Watson's lemma to be legitimately applied, we will find that f must be smooth with all derivatives of at most polynomial growth *and* at most polynomial *decay*, as $u \to +\infty$.

For simplicity *assume* that there is exactly *one* point u_o at which $f'(u_o) = 0$, and that $f''(u_o) > 0$. Further, assume that $f(u)$ goes to $+\infty$ at 0^+ and at $+\infty$. Since $f'(u) > 0$ for $u > u_o$ and $f'(u) < 0$ for $0 < u < u_o$, on each of these two intervals there is a smooth square root $\sqrt{f(u) - f(u_o)}$ and there are smooth functions F, G such that

$$\begin{cases} F(\sqrt{f(u) - f(u_o)}) \;=\; u & (\text{for } u_o < u < +\infty) \\[2ex] G(\sqrt{f(u) - f(u_o)}) \;=\; u & (\text{for } 0 < u < u_o) \end{cases}$$

Then

$$\int_0^\infty e^{-s\,f(u)}\,du \;=\; e^{-sf(u_o)} \int_0^{u_o} e^{-s\,(f(u)-f(u_o))}\,du + e^{-sf(u_o)} \int_{u_o}^\infty e^{-s\,(f(u)-f(u_o))}\,du$$

$$= e^{-sf(u_o)} \Big(\int_0^\infty e^{-s\,x^2}\,F'(x)\,dx + \int_0^\infty e^{-s\,x^2}\,G'(x)\,dx \Big)$$

by letting $x = \sqrt{f(u) - f(u_o)}$ in the two intervals. In both integrals, replacing x by $\sqrt{x}$ gives Watson's-lemma integrals

$$\int_0^\infty e^{-s\,f(u)}\,du \;=\; e^{-sf(u_o)} \Big(\int_0^\infty e^{-sx}\,\tfrac{1}{2}x^{1/2}\,F'(\sqrt{x})\,\frac{dx}{x} + \int_0^\infty e^{-sx}\,\tfrac{1}{2}x^{1/2}\,G'(\sqrt{x})\,\frac{dx}{x} \Big)$$

At this point the needed conditions on F, hence, on f, become clear: since F must be smooth with all derivatives of at most polynomial growth, direct chain-rule computations show that it suffices that no derivative of f increases *or decreases* faster than polynomially as $u \to +\infty$. The assumptions $f'(u_o) = 0$ and $f''(u_o) > 0$ assure that F has a Taylor series expansion near 0, giving a suitable expansion

$$\tfrac{1}{2}x^{1/2}F'(x) \;=\; \tfrac{1}{2}F'(0)x^{1/2} + \frac{\frac{1}{2}F^{(2)}(0)}{1!}x^{3/2} + \frac{\frac{1}{2}F^{(3)}(0)}{2!}x^{5/2} + \frac{\frac{1}{2}F^{(4)}(0)}{3!}x^{7/2} + \ldots \qquad (\text{small } x > 0)$$

From this, the main term of the Watson's lemma asymptotics for the integral involving F would be

$$\int_0^\infty e^{-sx}\,\tfrac{1}{2}x^{1/2}\,F'(\sqrt{x})\,\frac{dx}{x} \;\sim\; \frac{\Gamma(\frac{1}{2})\,F'(0)}{2}\cdot\frac{1}{\sqrt{s}}$$

To determine $F'(0)$, or any higher coefficients, from $F(x) = u$, we have $F'(x)\cdot\frac{dx}{du} = 1$. Since

$$x \;=\; \sqrt{f(u) - f(u_o)} \;=\; \sqrt{(u-u_o)^2\cdot\frac{f''(u_o)}{2!} + \ldots} \;=\; \sqrt{\frac{f''(u_o)}{2}}\cdot\Big((u-u_o) + \ldots\Big)$$

the derivative is

$$\frac{dx}{du} \;=\; \sqrt{\frac{f''(u_o)}{2}}\cdot\Big(1 + O(u-u_o)\Big)$$

Thus,

$$F'(x) \;=\; \frac{1}{\frac{dx}{du}} \;=\; \sqrt{\frac{2}{f''(u_o)}}\cdot\Big(1 + O(u-u_o)\Big)$$

which allows evaluation at $x = 0$, namely

$$F'(0) \;=\; \sqrt{\frac{2}{f''(u_o)}}$$

The same argument applied to G gives $G'(0) = F'(0)$. Thus,

$$\int_0^\infty e^{-s\,f(u)}\,du \;\sim\; e^{-sf(u_o)}\cdot\frac{\Gamma(\frac{1}{2})\cdot 2\cdot\sqrt{\frac{2}{f''(u_o)}}}{2\sqrt{s}} \;=\; e^{-sf(u_o)}\cdot\frac{\sqrt{2\pi}}{\sqrt{f''(u_o)}}\cdot\frac{1}{\sqrt{s}}$$

Last, we verify that this outcome is what would be obtained by replacing $f(u)$ by its quadratic approximation

$$f(u_o) + \frac{f''(0)}{2!}\cdot(u-u_o)^2$$

in the exponent in the original integral, integrated over the whole line. The latter would be

$$\int_{-\infty}^\infty e^{s\cdot\big(f(u_o)+\frac{1}{2}f''(u_o)(u-u_o)^2\big)}\,du \;=\; e^{sf(u_o)}\int_{-\infty}^\infty e^{s\cdot\frac{1}{2}f''(u_o)(u-u_o)^2}\,du \;=$$

$$= e^{sf(u_o)}\int_{-\infty}^\infty e^{s\cdot\frac{1}{2}f''(u_o)u^2}\,du \;=\; e^{sf(u_o)}\cdot\frac{\sqrt{\pi}}{\sqrt{\frac{1}{2}f''(u_o)}}\cdot\frac{1}{\sqrt{s}} \;=\; e^{sf(u_o)}\cdot\frac{\sqrt{2\pi}}{\sqrt{f''(u_o)}}\cdot\frac{1}{\sqrt{s}}$$

This does indeed agree. Last, verify that the integral of the exponentiated quadratic approximation over $(-\infty, 0]$ is of a lower order of magnitude. Indeed, for $u \le 0$ and $u_o > 0$ we have $(u-u_o)^2 \ge u^2 + u_o^2$, and $f''(u_o) < 0$ by assumption, so

$$e^{sf(u_o)}\int_{-\infty}^0 e^{s\cdot\big(\frac{1}{2}f''(u_o)(u-u_o)^2\big)}\,du \;\le\; e^{sf(u_o)}\cdot e^{s\cdot\frac{1}{2}f''(u_o)\cdot u_o^2}\int_{-\infty}^0 e^{s\cdot\frac{1}{2}f''(u_o)u^2}\,du$$

$$\leq\ e^{sf(u_o)}\cdot e^{s\cdot\frac{1}{2}f''(u_o)\cdot u_o^2}\int_{-\infty}^{\infty} e^{s\cdot\frac{1}{2}f''(u_o)u^2}\,du\ =\ e^{sf(u_o)}\cdot e^{s\cdot\frac{1}{2}f''(u_o)\cdot u_o^2}\cdot\frac{\sqrt{2\pi}}{\sqrt{f''(u_o)}}\cdot\frac{1}{\sqrt{s}}$$

Thus, the integral over $(-\infty,0]$ has an additional exponential decay by comparison to the integral over the whole line, so the leading-term of the asymptotics of the integral from 0 to $+\infty$ is the same as those of the integral from $-\infty$ to $+\infty$.

The case of $\Gamma(s)$ can be converted to this situation as follows. For *real* $s>0$, in the integral

$$s\cdot\Gamma(s)\ =\ \Gamma(s+1)\ =\ \int_0^\infty e^{-u}\,u^s\,du\ =\ \int_0^\infty e^{-u+s\log u}\,du$$

can replace u by su, to put the integral into the desired form

$$s\cdot\Gamma(s)\ =\ \int_0^\infty e^{-su+s\log u+s\log s}\,s\,du\ =\ s\cdot e^{s\log s}\int_0^\infty e^{-s(u+\log u)}\,du$$

For complex s with $\mathrm{Re}(s)>0$, both $s\cdot\Gamma(s)$ and the integral $s\cdot e^{s\log s}\int_0^\infty e^{-s(u+\log u)}\,du$ are holomorphic in s, and they *agree* for *real* s. Thus, by the identity principle, they are equal for $\mathrm{Re}(s)>0$.

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