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The product expansion $\Delta(z) = q\prod(1-q^n)^{24}$

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[Weil 1968] used a converse theorem to reprove the product expansion for the unique holomorphic cuspform Δ of weight 12 for $SL_2(\mathbb{Z})$:

$$\Delta(z) \;=\; e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi inz})^{24}$$

Weil's argument is not quite as simple as [Siegel 1954], but is perhaps more memorable and more interesting, in light of the more general discussion of *converse theorems* in [Weil 1967], in effect as plausibility checks on the Taniyama-Shimura conjecture that Hasse-Weil zeta functions of modular curves over $\mathbb{Q}$ are attached to holomorphic elliptic modular forms.

We reproduce Weil's argument, and give Siegel's in an appendix.

In fact, Weil's observation of the connection between a simple converse theorem and a product formula may be anomalous. A demonstrably more fruitful way to think about product expansions of automorphic forms is illustrated in [Borcherds 1995], [Borcherds 1995b], [Borcherds 1998]. More recently, [Zwegers 2001], [Zwegers 2002] put Ramanujan's mock-theta products into the context of automorphic forms. See [Zagier 2006-7] and [Ono 2010] for surveys of the latter developments.

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1. *Weil's proof*

We slightly rewrite [Weil 1968].

Consider the Dirichlet series [1]

$$L(s) \;=\; \zeta(s)\cdot\zeta(s+1) \;=\; \sum_{m,n}\frac{1}{m}\frac{1}{(mn)^s} \;=\; \sum_{n=1}^{\infty}\Big(\sum_{d|n}\frac{1}{d}\Big)\frac{1}{n^s}$$

The *completed* version

$$\Lambda(s) \;=\; (2\pi)^{-s}\,\Gamma(s)\,L(s)$$

has functional equation inherited from $\zeta(s)$:

$$\Lambda(-s) \;=\; \Lambda(s)$$

Noting that $\zeta(2)/2\pi = \pi/12$,

$$\Lambda(s) \;=\; \frac{\pi/12}{s-1} - \frac{1}{2s^2} - \frac{\pi/12}{s+1} + (\text{holomorphic})$$

The power series in $q = e^{2\pi iz}$ with the same coefficients is

$$F(z) \;=\; \sum_{m,n}\frac{1}{m}q^{mn} \;=\; \sum_{n}\Big(\sum_{m}\frac{1}{m}\,(q^n)^m\Big) \;=\; -\sum_{n}\log(1-q^n)$$

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[1] Weil surely was well aware that $L(s)$ is essentially the Mellin transform of the *constant term* (in the Laurent expansion in s) of the Eisenstein series E_s for $SL_2(\mathbb{Z})$ at $s=1$, where the Eisenstein series has a simple pole.

Recall Dedekind's eta-function

$$\eta(z) \;=\; q^{1/24} \prod_{n \geq 1} (1 - q^n)$$

Then

$$F(z) \;=\; -\Big(\frac{\log q}{24} + \sum_n \log(1-q^n)\Big) \;+\; \frac{\log q}{24} \;=\; -\log \eta \;+\; \frac{\pi i z}{12}$$

From the obvious Fourier-Mellin transform relation

$$\Lambda(s) \;=\; \int_0^\infty y^s \, F(iy) \, \frac{dy}{y} \qquad\qquad (\text{for } \mathrm{Re}\,(s) > 1)$$

Fourier-Mellin inversion gives

$$F(z) \;=\; \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} (z/i)^{-s} \, \Lambda(s) \, ds \qquad\qquad (\text{for } \sigma > 1)$$

Following Hecke and Weil, move the vertical line to $\mathrm{Re}\,(s) = -\sigma$, picking up residues at $1, 0, -1$:

$$F(z) \;=\; \int_{-\sigma - i\infty}^{-\sigma + i\infty} (z/i)^{-s} \, \Lambda(s) \, ds + \Big(\frac{\pi}{12} \cdot (z/i)^{-1} + \tfrac{1}{2} \log(z/i) - \frac{\pi}{12} \cdot (z/i)\Big)$$

$$= \int_{-\sigma - i\infty}^{-\sigma + i\infty} (z/i)^{-s} \, \Lambda(s) \, ds + \frac{\pi i}{12 z} + \tfrac{1}{2} \log(z/i) - \frac{\pi z}{12 i}$$

The functional equation $\Lambda(-s) = \Lambda(s)$ allows conversion of the integral on $\mathrm{Re}\,(s) = -\sigma$ into

$$\frac{1}{2\pi i} \int_{-\sigma - i\infty}^{-\sigma + i\infty} (z/i)^{-s} \, \Lambda(-s) \, ds \;=\; \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Big(\frac{-1}{z/i}\Big)^{-s} \, \Lambda(s) \, ds \;=\; F(-1/z)$$

That is,

$$F(z) \;=\; F(-1/z) + \frac{\pi i}{12 z} + \tfrac{1}{2} \log(z/i) - \frac{\pi z}{12 i}$$

Using $F(z) = \pi i z/12 - \log \eta(z)$, this is

$$\frac{\pi i z}{12} - \log \eta(z) \;=\; \frac{\pi i(-1/z)}{12} - \log \eta(-1/z) + \frac{\pi i}{12 z} + \tfrac{1}{2} \log(z/i) - \frac{\pi z}{12 i}$$

which simplifies to

$$\log \eta(z) \;=\; \log \eta(-1/z) - \tfrac{1}{2} \log(z/i)$$

Exponentiating and taking the 24^{th} power:

$$\eta^{24}(z) \;=\; \eta^{24}(-1/z) \cdot (z/i)^{-12}$$

or

$$\eta^{24}(-1/z) \;=\; z^{12} \cdot \eta^{24}(z)$$

That is, η^{24} has the two functional equations

$$\eta^{24}(z+1) \;=\; \eta^{24}(z) \qquad\qquad \eta^{24}(-1/z) \;=\; z^{12} \cdot \eta^{24}(z)$$

and goes to 0 as $\mathrm{Im}\,(z) \to +\infty$. Since $SL_2(\mathbb{Z})$ is generated by the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

giving the maps $z \to -1/z$ and $z \to z+1$, evidently η is a *holomorphic cuspform* of weight 12, with leading Fourier coefficient 1. Thus, it is $\Delta(z)$, and we have the product expansion of $\Delta(z)$:

$$\Delta(z) \;=\; \eta^{24}(z) \;=\; q\prod_{n\geq 1}(1-q^n) \;=\; e^{2\pi iz}\prod_{n\geq 1}(1-e^{2\pi inz})$$

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2. *Appendix: Siegel's proof*

Siegel's argument from 1954 is slightly simpler than Weil's from 1968. As expected, both make heavy use of various coincidences in this simple situation. We reproduce Siegel's argument without much rewriting, to capture its simple but *ad-hoc* flavor.

With η the 24^{th} root of Δ, with $q = e^{2\pi iz}$, taking a logarithm,

$$\frac{1}{12}\pi iz - \log\eta(z) \;=\; -\sum_{\ell=1}^{\infty}\log(1-q^\ell) \;=\; \sum_{k,\ell\geq 1}\frac{1}{k}q^{k\ell} \;=\; \sum_{k\geq 1}\frac{1}{k}\cdot\frac{q^k}{1-q^k} \;=\; \sum_{k\geq 1}\frac{1}{k}\cdot\frac{1}{q^{-k}-1}$$

This suggests proving the functional equation in the form

$$\pi i\frac{z+z^{-1}}{12} + \tfrac{1}{2}\log\frac{z}{i} \;=\; \sum_{k=1}^{\infty}\frac{1}{k}\Big(\frac{1}{e^{-2\pi ikz}-1} - \frac{1}{q^{-2\pi ik/z}-1}\Big)$$

Let

$$f(w) \;=\; \cot w\cdot\cot w/z$$

and let ν run over values $(n+\frac{1}{2})\pi$ for $0 \leq n \in \mathbb{Z}$. Then $f(\nu w)/w$ has *simple* poles at $w = \pm\pi k/\nu$ and at $w = \pm\pi kz/\nu$, with respective residues

$$\frac{1}{\pi k}\cot\frac{\pi k}{z} \qquad\text{and}\qquad \frac{1}{\pi k}\cot\pi kz \qquad\qquad (\text{for } k = 1,2,3,\ldots)$$

and a *triple* pole at at $w = 0$ with residue $-\frac{1}{3}(z+z^{-1})$.

Let γ be the path tracing counter-clockwise the outline of the parallelogram with vertices $1, z, -1, -z$. By residues,

$$\pi\frac{z+z^{-1}}{12} + \int_\gamma f(\nu w)\frac{dw}{8w} \;=\; \frac{i}{2}\sum_{k\geq 1}\frac{1}{k}(\cot\pi kz + \cot\pi k/z) \;=\; \sum_{k=1}^{\infty}\frac{1}{k}\Big(\frac{1}{q^{-k}-1} - \frac{1}{q^{-k/z}-1}\Big)$$

The parameters n or ν only appear in the contour integral on the left-hand side. To evaluate it, let as $n \to +infty$. In this limit, $f(\nu w)$ is uniformly bounded on γ, and has limiting values on the sides (excluding the vertices, where there are discontinuities) $1, -1, 1, -1$, respectively. The limit of the contour integral is

$$\int_\gamma f(\nu w)\frac{dw}{8w} \;=\; \Big(\int_1^z - \int_z^{-1} + \int_{-1}^{-z} - \int_{-z}^{1}\Big)\frac{dw}{w} \;=\; 4\,\log\frac{z}{i}$$

This gives the functional equation. ///

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