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Number theory exercises 07

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Due Fri, 27 Jan 2012, preferably as PDF emailed to me.

[number theory 07.1] Verify that $dg = dx\,\frac{dy}{y}$ in coordinates $g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$ is a right Haar measure on the group

$$G \;=\; \{\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \;:\; 0 < y \in \mathbb{R},\; x \in \mathbb{R}\}$$

[number theory 07.2] Verify that $dg = dx\,dy\,dz$ in coordinates $g = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$ is a right Haar measure on the group

$$G \;=\; \{\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \;:\; x, y, z \in \mathbb{R}\}$$

[number theory 07.3] Verify that $G = SL_2(k)$, the two-by-two matrices with entries in a field k with more than 2 elements, with determinant 1, has the property $G = [G, G]$.

Hint: To get started, note that

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} \;=\; \begin{pmatrix} 1 & a^2x \\ 0 & 1 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} \;=\; \begin{pmatrix} 1 & x(a^2-1) \\ 0 & 1 \end{pmatrix}$$

This shows that all unipotent upper-triangular matrices are commutators. Similarly for lower-triangular unipotent.
