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Number theory exercises 09

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Due Mon, 05 Mar 2012, preferably as PDF emailed to me.

[number theory 09.1] For $y \in \mathbb{Z}_p$, what are the eigenvalues and eigenvectors of the *unitary* operator $T : L^2(\mathbb{Z}_p) \to L^2(\mathbb{Z}_p)$ defined by $Tf(x) = f(x+y)$?

[number theory 09.2] With $G = \mathbb{Z}/n$, let $T : L^2(G) \to L^2(G)$ be the *first difference* operator

$$Tf(x) \;=\; f(x+1) - f(x)$$

Find the eigenvalues and eigenvectors of T.

[number theory 09.3] With $G = \mathbb{Z}/n$, let $T : L^2(G) \to L^2(G)$ be the *second difference* operator

$$Tf(x) \;=\; \big(f(x+1) - f(x)\big) - \big(f(x) - f(x-1)\big) \;=\; f(x+1) - 2f(x) + f(x-1)$$

Find the eigenvalues and eigenvectors of T.

[number theory 09.4] With $G = \mathbb{Z}/n \oplus \mathbb{Z}/n$, let $T : L^2(G) \to L^2(G)$ be the *discrete Laplacian*

$$Tf(x) \;=\; f(x+(1,0)) - 2f(x) + f(x+(1,0)) + f(x+(0,1)) - 2f(x) + f(x+(0,1))$$

Find the eigenvalues and eigenvectors of T.

[number theory 09.5] What are the eigenvalues and eigenvectors of the *compact self-adjoint* operator $T : L^2(\mathbb{Z}_p) \to L^2(\mathbb{Z}_p)$ defined by

$$Tf(x) \;=\; \int_{\mathbb{Z}_p} |x-y| \cdot f(y)\, dy$$

[number theory 09.6] * *[Starred problems are optional]* Define a Hilbert-Schmidt operator $T : L^2[0,1] \to L^2[0,1]$ by the continuous kernel

$$K(x,y) \;=\; \begin{cases} (x-1)\cdot y & (\text{for } 0 \le y < x) \\ \\ x \cdot (y-1) & (\text{for } x < y \le 1) \end{cases}$$

Find mild hypotheses on f so that the function $u = Tf$ solves the differential equation $u'' = f$. That is, the differential operator d^2/dx^2 has a (one-sided) inverse which is a compact operator.

[number theory 09.7] * *[Starred problems are optional]* Show that *Volterera operator* $V : L^2[0,1] \to L^2[0,1]$ given by

$$Vf(x) \;=\; \int_0^x f(t)\; dt \qquad\qquad (\text{for } f \in L^2[0,1])$$

is *compact*, although *not* self-adjoint. Show that $\lim_{n\to\infty} |V^n| = 0$, and that V has *no* eigenvectors.
