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Number theory exercises 02

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Due Fri, 30 Sept 2011, preferably as PDF emailed to me.

[number theory 02.1] Show that the *ideal* norm and *Galois* norm agree on $\mathbb{Z}[i]$. That is, show that for $0 \neq \alpha \in \mathbb{Z}[i]$,

$$\mathrm{card}\, \mathbb{Z}[i]/\big(\alpha \cdot \mathbb{Z}[i]\big) \;=\; \alpha \cdot \overline{\alpha}$$

[number theory 02.2] Show that in a PID every non-zero prime ideal is maximal.

[number theory 02.3] Carefully show that for a, b in a commutative ring R, with $\overline{a}$ the image of a in $R/\langle b\rangle$ and $\overline{b}$ the image of b in $R/\langle a\rangle$, there is a natural isomorphism

$$(R/\langle a\rangle)/\langle\overline{b}\rangle \;=\; (R/\langle b\rangle)/\langle\overline{a}\rangle$$

[number theory 02.4] For rational $p > 2$ splitting in $\mathbb{Z}[i]$, and for ρ any representative in $\mathbb{Z}$ for a square root of -1 mod p, show that the pairs $p, \rho - i$ and $p, \rho + i$ generate the two prime ideals into which $p \cdot \mathbb{Z}[i]$ factors.

[number theory 02.5] Show that $\mathbb{Z}[\sqrt{2}]$ is Euclidean.
