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Number theory exercises 03

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Due Mon, 10 Oct 2011, preferably as PDF emailed to me.

[number theory 03.1] Prove that $\sqrt{-1}$ exists in $\mathbb{Q}_5$.

[number theory 03.2] Prove that a primitive 11^{th} root of unity exists in $\mathbb{Q}_{23}$.

[number theory 03.3] Prove that addition, multiplication, and inversion (away from 0) are *continuous* on $\mathbb{Q}_p$.

[number theory 03.4] Show that the usual power series $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots$ converges in $\mathbb{Q}_p$ for $|x| < \frac{1}{p-1}$. (*Hint:* First show that the power p^ℓ of p dividing $n!$ is bounded by

$$\ell \;\le\; \frac{n}{p}+\frac{n}{p^2}+\frac{n}{p^3}+\ldots$$

That is, there are at most n/p integers less than n and divisible by p, there are at most n/p^2 numbers less than n and divisible by p^2, ...)

[number theory 03.5] * (Starred problems are optional.) Show that there are only finitely-many quadratic extensions of $\mathbb{Q}_p$. In fact, for p odd, there are exactly *three*, while there are exactly 7 quadratic extensions of $\mathbb{Q}_2$.
