(*November 10, 2011*)

Number theory exercises 04

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

Due Wed, 26 Oct 2011, preferably as PDF emailed to me.

[number theory 04.1] *Localization.* Let S be a *multiplicative subset* of a commutative ring R, that is, S is closed under multiplication. To avoid silliness, assume $0 \notin S$, and that R has a unit 1. The *localization* $S^{-1}R$ of R at S can be *characterized* as a ring S^{-1} with a ring hom $j : R \to S^{-1}R$ such that $j(S) \subset (R')^\times$, such that for *any* ring hom $f : R \to X$ with $f(S) \subset X^\times$, there is a unique ring hom $F : S^{-1}R \to X$ so that $f = F \circ j$:

$$\begin{tikzcd} S^{-1}R \arrow[dr, dashed, "F"] & \\ R \arrow[u, "j"] \arrow[r, "f"] & X \end{tikzcd}$$

The usual categorical argument proves uniqueness up to unique isomorphism. Show that the following *construction* yields $j : R \to S^{-1}R$. Put an equivalence relation $\sim$ on ordered pairs (r, s) with $r \in R$ and $s \in S$ by $(r, s) \sim (r', s')$ if there is $s'' \in S$ such that $s''(rs' - r's) = 0$. Let $j(r) = (r, 1)$.

When R is a domain, the simpler requirement $rs' = r's$ suffices.

[number theory 04.2] For a non-zero prime ideal $\mathfrak{p}$ of R, the localization with respect to the multiplicative subset $S = R - \mathfrak{p}$ is often denoted $R_{(\mathfrak{p})}$. (Or even $R_\mathfrak{p}$, but the latter might better be reserved for a *completion.*) Show that $R_{(\mathfrak{p})}$ has a *unique maximal ideal*, consisting of (equivalence classes of) (r, s) with $r \in \mathfrak{p}$ and $s \in S$.

[number theory 04.3] Show that the set of proper ideals of $S^{-1}R$ *injects* to the set of ideals of R not meeting S.

[number theory 04.4] Show that $T^5 - XT + X$ factors into linear factors in $\mathbb{C}[[X^{1/5}]][T]$, while $T^5 - XT^2 + X^2$ factors into linear factors in $\mathbb{C}[[X^{1/6}]][T]$.

[number theory 04.5] Show that the Galois group of $\mathbb{C}(X^{1/n})$ over $\mathbb{C}(X)$ is naturally isomorphic to $\mathbb{Z}/n$ with addition.

[number theory 04.6] Show that the Galois group of $\mathbb{C}((X^{1/n}))$ over $\mathbb{C}((X))$ is naturally isomorphic to $\mathbb{Z}/n$ with addition.

[number theory 04.7] * (Starred problems are optional.) Show that the Galois group of an algebraic closure $\overline{\mathbb{C}((X))}$ of $\mathbb{C}((X))$, over $\mathbb{C}((X))$, is naturally isomorphic to

$$\widehat{\mathbb{Z}} = \lim_n \mathbb{Z}/n \approx \prod_p \mathbb{Z}_p$$
