Context: Finiteness of class number and *Dirichlet's units theorem* are corollaries of *Fujisaki's lemma*, that $\mathbb{J}^1/k^\times$ is *compact.*

... corollary of existence and uniqueness of Haar measure on $\mathbb{A}$ and $\mathbb{A}/k$... and compactness of $\mathbb{A}/k$, so that it has finite invariant measure. We verified this compactness.

Change-of-measure: for *idele* α, the change-of-measure on $\mathbb{A}$ is

$$\frac{\text{meas}\,(\alpha E)}{\text{meas}\,(E)} = |\alpha| \qquad\qquad (\text{for measurable } E \subset \mathbb{A})$$

The latter is the full version of a familiar fact on the real line: for $\alpha \in \mathbb{R}^\times$ and measurable $E \subset \mathbb{R}$, written exactly the same way

$$\frac{\text{meas}\,(\alpha E)}{\text{meas}\,(E)} = |\alpha| \qquad\qquad (\text{for measurable } E \subset \mathbb{R})$$

Local version for p-adic completions, first.

Measure on $\mathbb{Q}_p$?

We want the *regularity* promised by Riesz on locally compact, Hausdorff, countably-based topological spaces: the measure of a set is the *inf* of measure of opens containing it, and *sup* of measure of compacts contained in it.

For (locally compact...) *totally disconnected abelian groups* such as $\mathbb{Q}_p$, there is a local basis $U_n = p^n\mathbb{Z}_p$ at 0 consisting of open *subgroups*. Since $\mathbb{Z}_p$ is also *compact* (and closed), so is each U_n. Since $\mathbb{Z}_p$ is the disjoint union

$$\mathbb{Z}_p \;=\; p^n\mathbb{Z}_p \sqcup (1+p^n\mathbb{Z}_p) \sqcup (2+p^n\mathbb{Z}_p) \sqcup \ldots \sqcup ((p^n-1)+p^n\mathbb{Z}_p)$$

of p^n of these cosets, by additivity

$$\text{meas}\,(p^n\mathbb{Z}_p) \;=\; p^{-n}\cdot \text{meas}\,(\mathbb{Z}_p)$$

Normalizing $\text{meas}\,(\mathbb{Z}_p) = 1$ specifies a regular measure on $\mathbb{Q}_p$.

Totally disconnected spaces have the advantage that many *simple functions* (meaning assuming only finitely-many values) are *continuous*, because many nice open sets are also closed:

$$p^n\mathbb{Z}_p \;=\; \{x \in \mathbb{Q}_p : |x|_p < \frac{1}{p^{n+1}}\} \;=\; \{x \in \mathbb{Q}_p : |x|_p \le \frac{1}{p^n}\}$$

Since addition is continuous, $x \to x + y$ is a homeomorphism of $\mathbb{Q}_p$ to itself, so $p^n\mathbb{Z}_p + y$ is both open and closed.

Claim: every $f \in C^o_c(\mathbb{Q}_p)$ can be *approximated* by finite linear combinations of characteristic functions of sets $p^n\mathbb{Z}_p + y$.

Remark: The appropriate topology on $C^o_c(\mathbb{Q}_p)$, or on $C^o_c(\mathbb{R})$, is *not* sup-norm. But each subspace $C^o_c(p^{-k}\mathbb{Z}_p)$ *is* topologized by sup-norm, and is *complete metric.*

The topology on the whole space $C^o_c(\mathbb{Q}_p)$ is the *colimit* of the spaces $C^o_c(p^{-k}\mathbb{Z}_p)$. It is (quasi-) complete, but is not complete metric, since it violates the conclusion of Baire category, namely, it is a countable union of nowhere-dense subsets.

Proof of claim: Since all the sets $p^{-k}\mathbb{Z}_p$ are homeomorphic, without loss of generality take $k = 0$. Let $f \in C_c^o(\mathbb{Z}_p)$. Fix $\varepsilon > 0$. For each $x \in \mathbb{Z}_p$, let $U_{n_x} = p^{n_x}\mathbb{Z}_p$ be a small-enough neighborhood of 0 so that $|f(x) - f(x')| < \varepsilon$ for $x' \in x + U_{n_x}$.

By compactness of $\mathbb{Z}_p$, there is a finite subcover $x_i + U_i$ of $\mathbb{Z}_p$. Let $U = \bigcap_i U_i$. The intersection is finite, so is open. We *claim* that for $x, x' \in \mathbb{Z}_p$ with $x - x' \in U$, necessarily $|f(x) - f(x')| < 2\varepsilon$. To see this, let $x \in x_i + U_i$. Then

$$x' \ \in \ x + U \ \subset \ (x_i + U_i) + U \ = \ x_i + (U_i + U) \ = \ x_i + U_i$$

As U is a sub*group*, $\mathbb{Z}_p$ is a finite *disjoint* union of cosets $U + y$. Define a simple function

$$\varphi(x) \ = \ f(y) \text{ for } x \in U + y$$

This differs from f by at most 2ε. ///

The continuity of these simple functions allows definition of integrals of $C_c^o(\mathbb{Q}_p)$ functions without going outside $C_c^o(\mathbb{Q}_p)$, by taking continuous simple function φ approximating f within ε, and

$$\int_{\mathbb{Q}_p} f \;=\; \lim_{\varepsilon\to 0} \sum_y \varphi(y)\cdot \mathrm{meas}\,(U+y) \;=\; \lim_{\varepsilon\to 0} \mathrm{meas}\,(U)\cdot \sum_y \varphi(y)$$

Let $S(\varphi) = \sum_y \varphi(y)\cdot \mathrm{meas}\,(U)$, noting that this does depend on the finite cover by U-cosets.

It is not surprising that the limit is *well-defined*, much as Riemann sums approximating integrals of continuous functions on $\mathbb{R}$ give a well-defined limit: given simple φ, ψ approximating f within ε,

$$|S(\varphi) - S(\psi)| \;<\; 2\cdot\varepsilon\cdot \mathrm{meas}\,(\mathrm{spt} f)$$

Thus, the sums $S(\varphi)$ are a *Cauchy net* of complex numbers, proving the well-definedness.

Translation-invariance: Again, take advantage of the total-disconnectedness.

Given $f \in C^o_c(\mathbb{Q}_p)$ and $g \in \mathbb{Q}_p$, let $k \in \mathbb{Z}$ be large enough so that $p^{-k}\mathbb{Z}_p$ contains both spt$f$ and g. Then $x \to f(x+g)$ also has support inside $p^{-k}\mathbb{Z}_p$.

For a simple function φ approximating f, with φ a linear combination of characteristic functions of cosets $U + y$, $x \to x + g$ simply permutes these cosets. Thus,

$$\sum_y \varphi(y) \cdot \mathrm{meas}\,(U) \;=\; \sum_y \varphi(y+g) \cdot \mathrm{meas}\,(U+g)$$

Thus, $\int_{\mathbb{Q}_p} f(x+g)\,dx \;=\; \int_{\mathbb{Q}_p} f(x)\,dx$. ///

Uniqueness!?! Does taking meas$(\mathbb{Z}_p) = 1$ and the above construction of an integral give the only possible invariant integral/measure on $\mathbb{Q}_p$?

Temporarily ignoring any general assertion of uniqueness of Haar measure, let's take advantage of the special features here:

$\mathbb{Z}_p$ is *open*, so is measurable. It is compact, so its measure is *finite*. Thus, we can renormalize a given Haar measure μ so that $\mu(\mathbb{Z}_p) = 1$.

Since $\mathbb{Z}_p$ is a disjoint union of p^n *translates* of $p^n\mathbb{Z}_p$, all with the same measure, by translation-invariance. Thus, $\mu(p^n\mathbb{Z}_p) = p^{-n}$. Thus, integrals of simple functions are completely determined.

We saw that each $C_c^o(p^{-k}\mathbb{Z}_p)$ can be approximated by simple functions. By the required positivity/continuity of the invariant integral, this determines integrals of $C_c^o(\mathbb{Q}_p)$ completely. ///

Change-of-measure: We probably believe the assertion

$$\frac{\text{meas}\,(\alpha E)}{\text{meas}\,(E)} = |\alpha| \qquad\qquad (\alpha \in \mathbb{R}^\times, \text{ measurable } E \subset \mathbb{R})$$

Before considering the p-adic case, the *complex* claim is

$$\frac{\text{meas}\,(\alpha E)}{\text{meas}\,(E)} = |\alpha|_{\mathbb{C}} \;=\; |\alpha|^2 \qquad\qquad (\alpha \in \mathbb{C}^\times, \text{ measurable } E \subset \mathbb{C})$$

Recall that the product-formula normalization of the norm on $\mathbb{C}$ is $|\alpha|_{\mathbb{C}} = |N_{\mathbb{R}}^{\mathbb{C}}\alpha|_{\mathbb{R}}$, giving the square of the *extension* normalization. We can use the usual parallelogram argument: take $\mathbb{R}$ basis $1, i$ of $\mathbb{C}$, and $\alpha = a + bi$. Then $\alpha \cdot 1 = a + bi$ and $\alpha \cdot i = -b + a$, and

$$\left| \det \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \right| \;=\; a^2 + b^2 \;=\; |\alpha|^2 \;=\; |\alpha|_{\mathbb{C}}$$

p-adic change-of-measure: Take advantage of the special nature of things here: for $\alpha \in \mathbb{Q}^p$ with $\alpha = p^{-k} \cdot \eta$ with $\eta \in \mathbb{Z}_p$ and $k \geq 0$, the set $\alpha \cdot \mathbb{Z}_p = p^{-k}\mathbb{Z}_p$ is a disjoint union of p^k copies of $\mathbb{Z}_p$, so has measure p^k. Oppositely, for $\alpha = p^k \cdot \eta$ with $k \geq 0$, $\mathbb{Z}_p$ is a disjoint union of p^k copies of $\alpha \cdot \mathbb{Z}_p$, so the measure of $\alpha \cdot \mathbb{Z}_p$ is p^{-k}.

In both cases, we have

$$\frac{\text{meas}\,(\alpha \cdot \mathbb{Z}_p)}{\text{meas}\,(\mathbb{Z}_p)} \;=\; |\alpha|_p$$

Since $\nu(E) = \text{meas}\,(\alpha \cdot E)$ is another translation-invariant measure, by uniqueness it is a constant multiple of our constructed measure. We need only determine the constant, and computing the measure of $\alpha \cdot \mathbb{Z}_p$ does this. ///

Change-of-measure: adeles

The rational adele group is not a product, but it is an ascending union (colimit) of products

$$\mathbb{A}_S \;=\; \mathbb{R} \times \prod_{v \in S} \mathbb{Q}_v \times \prod_{v \notin S} \mathbb{Z}_v$$

over finite sets S of places outside of which elements are locally integral.

Countable products of *countably-based* locally-compact spaces with regular Borel measures have well-behaved product measures specified (up to completion, which is irrelevant for us) by the measures on the factors: the product topology has a countable basis, and any open is a countable union of basis opens, so measures of opens are completely determined.

Group invariance is inherited by the product measure.