Next: *harmonic analysis*, on $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$, $\mathbb{Q}_p$, $\mathbb{A}$, and $\mathbb{A}_k/k$, are the key ingredients in Iwasawa-Tate's 1950 modernization of Hecke's 1918-20 proof of continuation and functional equation of zeta functions of all number fields, and *all* L-functions for $GL(1)$.

Riemann's (1857-8) treatment of $\zeta_{\mathbb{Q}}(s)$ suffices for Dirichlet L-functions over $\mathbb{Q}$, and complex quadratic extensions of $\mathbb{Q}$. Reciprocity laws reduce factor zetas of abelian extensions of $\mathbb{Q}$ into Dirichlet L-functions over $\mathbb{Q}$.

Dedekind ($\sim$1870) meromorphically continued zetas of number fields to small neighborhoods of $s = 1$, but this is insufficient.

Hecke's 1918-20 proofs used Poisson summation for $\mathfrak{o} \subset k \otimes_{\mathbb{Q}} \mathbb{R}$. Iwasawa-Tate used the Weil-Pontryagin-Godement harmonic analysis on abelian topological groups, and everything became much clearer, and more memorable.

We need the abelian topological group analogue of *characters* $x \to e^{2\pi i x\xi}$ for $\xi \in \mathbb{R}$, on $\mathbb{R}$, and *Fourier transforms*

$$\widehat{f}(\xi) \;=\; \mathscr{F} f(\xi) \;=\; \int_{\mathbb{R}} e^{-2\pi i x\xi}\, f(x)\, dx$$

and *inversion*

$$f(x) \;=\; \mathscr{F}^{-1} \widehat{f}(x) \;=\; \int_{\mathbb{R}} e^{2\pi i \xi x}\, \widehat{f}(\xi)\, d\xi$$

for nice functions f on $\mathbb{Q}_p$ and $\mathbb{A}$. Similarly for all completions k_v and adeles $\mathbb{A}_k$ of number fields. And *adelic Poisson summation*

$$\sum_{x\in k} f(x) \;=\; \sum_{x\in k} \mathscr{F} f(x) \qquad\qquad \text{(for suitable } f \text{ on } \mathbb{A}_k)$$

Fujisaki's lemma packs up the Units Theorem and finiteness of class groups exactly as needed by Iwasawa-Tate.

After these preparations, the argument will be identical to Riemann's!

Bibliographic and historical notes:

[Iwasawa 1950/52] K. Iwasawa, [brief announcement], in Proceedings of the 1950 International Congress of Mathematicians, Vol. 1, Cambridge, MA, 1950, p. 322, Amer. Math. Soc., Providence, RI, 1952.

[Iwasawa 1952/92] K. Iwasawa, *Letter to J. Dieudonné*, dated April 8, 1952, in *Zeta Functions in Geometry*, editors N. Kurokawa and T. Sunada, Adv. Studies in Pure Math. **21** (1992), 445-450.

[Tate 1950/67] J. Tate, *Fourier analysis in number fields and Hecke's zeta functions*, Ph.D. thesis, Princeton (1950), in *Alg. No. Theory*, J. Cassels and J. Frölich, ed., Thompson Book Co., 1967.

The latter was not public until 1967, although because it was written out in great detail, received much more publicity than Iwasawa's ICM announcement. That story has resulted in these ideas often being mis-labelled *Tate's thesis.* It is better to refer to these ideas as *Iwasawa-Tate theory.*

Iwasawa-Tate was not just a rewrite: made the general case an obvious parallel to Riemann's. Further, it was a prototype for Gelfand-Piatetski-Shapiro's (1963) and Jacquet-Langlands' (1971) modernization of the classical theory of L-functions attached to $GL(2)$. This paved the way for Langlands' program's conception, and the Jacquet-Shalike-PiatetskiShapiro development of automorphic forms and L-functions on $GL(n)$.

[Selberg 1956] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric spaces, with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47-87

The latter gave yet-another impetus to the emerging viewpoint that the discussion of zeta functions and L-functions, which had appeared from 1830's through 1930's to be a conglomeration of *ad hoc* manipulations of integrals and sums, instead was a manifestation of far more structure: harmonic analysis, representation theory, and spectral theory of operators.

No small subgroups:

The circle group S^1 *has no small subgroups*, in the sense that there is a neighborhood U of the identity $1 \in S^1$ such that the only subgroup of S^1 inside U is the trivial group $\{1\}$.

Essentially the same proof works for *real Lie groups.*

Proof: Use the copy of S^1 inside the complex plane. We claim that taking

$$U = S^1 \cap \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$$

suffices: the only subgroup G of S^1 inside this U is $G = \{1\}$. Suppose not. Let $1 \neq e^{i\theta} \in G \cap U$. We can take $0 < \theta < \pi/2$, since both $\pm\theta$ must appear. Let $0 < \ell \in \mathbb{Z}$ be the smallest such that $\ell \cdot \theta > \pi/2$. Then, since $(\ell - 1) \cdot \theta < \pi/2$ and $0 < \theta < \pi/2$,

$$\frac{\pi}{2} < \ell \cdot \theta = (\ell - 1) \cdot \theta + \theta < \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Thus, $\ell \cdot \theta$ falls outside U, contradiction. ///

Unitary duals of abelian topological groups: For an abelian topological group G, the unitary dual $G^\vee$ is the collection of continuous group homomorphisms of G to the unit circle in $\mathbb{C}^\times$. For example, $\mathbb{R}^\vee \approx \mathbb{R}$, by $\xi \to (x \to e^{i\xi x})$.

Claim: $\mathbb{Q}_p^\vee \approx \mathbb{Q}_p$ and $\mathbb{A}^\vee \approx \mathbb{A}$.

Since $\mathbb{C}^\times$ contains no *small subgroups*, and since $\mathbb{Q}_p$ is a union of *compact* subgroups $p^{-k}\mathbb{Z}_p$, every element of $\mathbb{Q}_p^\vee$ has image in roots of unity in $\mathbb{C}^\times$, identified with $\mathbb{Q}/\mathbb{Z}$, so

$$\mathbb{Q}_p^\vee \;\approx\; \mathrm{Hom}^o(\mathbb{Q}_p, \mathbb{Q}/\mathbb{Z}) \qquad\qquad \text{(continuous homomorphisms)}$$

where $\mathbb{Q}/\mathbb{Z} \;=\; \mathrm{colim}\, \frac{1}{N}\mathbb{Z}/\mathbb{Z}$ is *discrete*. As a topological group, $\mathbb{Z}_p \;=\; \lim \mathbb{Z}/p^\ell\mathbb{Z}$, and $\mathbb{Z}_p$ is also a limit of the corresponding quotients of *itself*, namely,

$$\mathbb{Z}_p \;\approx\; \lim \mathbb{Z}_p/p^\ell\mathbb{Z}_p$$

Generally, any *abelian, totally disconnected* topological group G is such a limit of quotients:

$$G \approx \lim_K G/K \qquad\qquad (K \text{ compact open subgroup})$$

We do not need the general case, but its proof is not difficult. As a topological group,

$$\mathbb{Q}_p = \bigcup \frac{1}{p^\ell}\mathbb{Z}_p = \operatorname{colim} \frac{1}{p^\ell}\mathbb{Z}_p$$

Because of the *no small subgroups* property of the unit circle in $\mathbb{C}^\times$, every continuous element of $\mathbb{Z}_p^\vee = (\lim \mathbb{Z}/p^n\mathbb{Z})^\vee$ factors through some limitand

$$\mathbb{Z}_p/p^\ell\mathbb{Z}_p \approx \mathbb{Z}/p^\ell\mathbb{Z}$$

Thus,

$$\mathbb{Z}_p^\vee \;=\; \mathrm{colim}\,\Big(\mathbb{Z}_p/p^\ell\mathbb{Z}_p\Big)^\vee \;=\; \mathrm{colim}\,\frac{1}{p^\ell}\mathbb{Z}_p/\mathbb{Z}_p$$

since $\frac{1}{p^\ell}\mathbb{Z}_p/\mathbb{Z}_p$ is the dual to $\mathbb{Z}_p/p^\ell\mathbb{Z}_p$ under the pairing

$$\frac{1}{p^\ell}\mathbb{Z}_p/\mathbb{Z}_p \;\times\; \mathbb{Z}_p/p^\ell\mathbb{Z}_p \;\approx\; \frac{1}{p^\ell}\mathbb{Z}/\mathbb{Z} \;\times\; \mathbb{Z}/p^\ell\mathbb{Z}$$

by

$$\big(\frac{x}{p^\ell}+\mathbb{Z}\big) \;\times\; \big(y+p^\ell\mathbb{Z}\big) \;\longrightarrow\; xy+\mathbb{Z} \;\in\; \mathbb{Q}/\mathbb{Z}$$

The transition maps in the colimit expression for $\mathbb{Z}_p^\vee$ are inclusions, so

$$\mathbb{Z}_p^\vee \;=\; \mathrm{colim}\,\frac{1}{p^\ell}\mathbb{Z}_p/\mathbb{Z}_p \;\approx\; \Big(\mathrm{colim}\,\frac{1}{p^\ell}\mathbb{Z}_p\Big)/\mathbb{Z}_p \;\approx\; \mathbb{Q}_p/\mathbb{Z}_p$$

Thus,

$$\mathbb{Q}_p^\vee \;=\; \Big(\mathrm{colim}\,\frac{1}{p^\ell}\mathbb{Z}_p\Big)^\vee \;=\; \lim \frac{1}{p^\ell}\mathbb{Z}_p^\vee$$

As a topological group, $\frac{1}{p^\ell}\mathbb{Z}_p \;\approx\; \mathbb{Z}_p$ by multiplying by p^ℓ, so the dual of $\frac{1}{p^\ell}\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}_p^\vee \;\approx\; \mathbb{Q}_p/\mathbb{Z}_p$. However, the inclusions for varying ℓ are not the identity map, so for compatibility take

$$\Big(\frac{1}{p^\ell}\mathbb{Z}_p\Big)^\vee \;=\; \mathbb{Q}_p/p^\ell\mathbb{Z}_p$$

Thus,

$$\mathbb{Q}_p^\vee \;=\; \lim \mathbb{Q}_p/p^\ell\mathbb{Z}_p \;\approx\; \mathbb{Q}_p$$

because, $\mathbb{Q}_p$ is the projective limit of its quotients by compact open subgroups. ///

Claim: Both $\mathbb{A}^\vee \approx \mathbb{A}$ and $\mathbb{A}_{\rm fin}^\vee \approx \mathbb{A}_{\rm fin}$.

Proof: The same argument applies to $\widehat{\mathbb{Z}} = \lim \mathbb{Z}/N\mathbb{Z}$ and finite adeles $\mathbb{A}_{\rm fin} = \mathrm{colim}\, \frac{1}{N}\widehat{\mathbb{Z}}$, proving the self-duality of $\mathbb{A}_{\rm fin}$. Then the self-duality of $\mathbb{R}$ gives the self-duality of $\mathbb{A}$. ///

Remark: $\widehat{\mathbb{Z}}$ does also refer to $\mathrm{Hom}^o(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$, but needs to be *topologized* by the compact-open topology [below].

Remark: Nearly the same argument applies for an arbitrary finite extension k of $\mathbb{Q}$.

Corollary: Given *non-trivial* $\psi \in \mathbb{Q}_p^\vee$, every other element of $\mathbb{Q}_p^\vee$ is of the form $x \to \psi(\xi \cdot x)$ for some $\xi \in \mathbb{Q}_p$. Similarly, given *non-trivial* $\psi \in \mathbb{A}^\vee$, every other element of $\mathbb{A}^\vee$ is of the form $x \to \psi(\xi \cdot x)$ for some $\xi \in \mathbb{A}$. [below]

Remark: This sort of result is already familiar from the analogue for $\mathbb{R}$, that $x \to e^{i\xi x}$ for $\xi \in \mathbb{R}$ are all the unitary characters of $\mathbb{R}$.

Compact-discrete duality

For abelian topological groups G, pointwise multiplication makes $\widehat{G}$ an abelian group. A reasonable topology on $\widehat{G}$ is the *compact-open* topology, with a sub-basis

$$U = U_{C,E} = \{f \in \widehat{G} : f(C) \subset E\}$$

for compact $C \subset G$, open $E \subset S^1$.

Remark: The reasonable-ness of this topology is utilitarian. For a compact topological space X, $C^o(X)$ with the *sup-norm* is a *Banach space* . On non-compact X, the semi-norms given by *sups on compacts* make $C^o(X)$ a *Fréchet space* (assuming σ-compactness). The compact-open topology is the analogue for $C^o(X, Y)$ where the target Y is not normed. When X, Y are topological groups, the continuous functions $f : X \to Y$ consisting of *group homomorphisms* is a (locally compact, Hausdorff) topological group. [Later]

Granting for now that the compact-open topology makes $\widehat{G}$ an abelian (locally-compact, Hausdorf) topological group,

Theorem: The unitary dual of a *compact* abelian group is *discrete*. The unitary dual of a *discrete* abelian group is *compact*.

Proof: Let G be compact. Let E be a small-enough open in S^1 so that E contains no non-trivial subgroups of G. Using the compactness of G itself, let $U \subset \widehat{G}$ be the open

$$U = \{f \in \widehat{G} : f(G) \subset E\}$$

Since E is *small*, $f(G) = \{1\}$. That is, f is the trivial homomorphism. This proves discreteness of $\widehat{G}$ for compact G.

For G discrete, *every* group homomorphism to S^1 is continuous. The space of *all* functions $G \to S^1$ is the cartesian product of copies of S^1 indexed by G. By Tychonoff's theorem, this product is *compact*. For *discrete* X, the compact-open topology on the space $C^o(X, Y)$ of continuous functions from $X \to Y$ *is* the product topology on copies of Y indexed by X.

The set of functions f satisfying the group homomorphism condition

$$f(gh) = f(g) \cdot f(h) \qquad \text{(for } g, h \in G\text{)}$$

is *closed*, since the group multiplication $f(g) \times f(h) \to f(g) \cdot f(h)$ in S^1 is continuous. Since the product is also *Hausdorff*, $\widehat{G}$ is also compact. ///

Theorem: $(\mathbb{A}/k)\widehat{\ } \approx k$. In particular, given any non-trivial character ψ on $\mathbb{A}/k$, *all* characters on $\mathbb{A}/k$ are of the form $x \to \psi(\alpha \cdot x)$ for some $\alpha \in k$.

Proof: For a (discretely topologized) number field k with adeles $\mathbb{A}$, $\mathbb{A}/k$ is *compact*, and $\mathbb{A}$ is *self-dual*.

Because $\mathbb{A}/k$ is compact, $(\mathbb{A}/k)\widehat{\ }$ is *discrete*. Since multiplication by elements of k respects cosets $x + k$ in $\mathbb{A}/k$, the unitary dual has a k-vectorspace structure given by

$$(\alpha \cdot \psi)(x) = \psi(\alpha \cdot x) \qquad\qquad (\text{for } \alpha \in k,\ x \in \mathbb{A}/k)$$

There is no topological issue in this k-vectorspace structure, because $(\mathbb{A}/k)\widehat{\ }$ is discrete. The quotient map $\mathbb{A} \to \mathbb{A}/k$ gives a natural *injection* $(\mathbb{A}/k)\widehat{\ } \to \widehat{\mathbb{A}}$.

Given non-trivial $\psi \in (\mathbb{A}/k)\widehat{}$, the k-vectorspace $k \cdot \psi$ inside $(\mathbb{A}/k)\widehat{}$ injects to a copy of $k \cdot \psi$ inside $\widehat{\mathbb{A}} \approx \mathbb{A}$. *Assuming* for a moment that the image in $\mathbb{A}$ is essentially the same as the diagonal copy of k, $(\mathbb{A}/k)\widehat{}/k$ injects to $\mathbb{A}/k$. The topology of $(\mathbb{A}/k)\widehat{}$ is discrete, and the quotient $(\mathbb{A}/k)\widehat{}/k$ is still discrete. These maps are continuous group homs, so the image of $(\mathbb{A}/k)\widehat{}/k$ in $\mathbb{A}/k$ is a discrete subgroup of a compact group, so is *finite*. Since $(\mathbb{A}/k)\widehat{}$ is a k-vectorspace, $(\mathbb{A}/k)\widehat{}/k$ is a singleton. Thus, $(\mathbb{A}/k)\widehat{} \approx k$, if the image of $k \cdot \psi$ in $\mathbb{A} \approx \widehat{\mathbb{A}}$ is the usual diagonal copy.

To see how $k \cdot \psi$ is imbedded in $\mathbb{A} \approx \widehat{\mathbb{A}}$, fix non-trivial ψ on $\mathbb{A}/k$, and let ψ be the induced character on $\mathbb{A}$. The self-duality of $\mathbb{A}$ is that the action of $\mathbb{A}$ on $\widehat{\mathbb{A}}$ by $(x \cdot \psi)(y) = \psi(xy)$ gives an *isomorphism*. The subgroup $x \cdot \psi$ with $x \in k$ is certainly the usual diagonal copy. ///

Next: Fourier transforms, Fourier inversion, Schwartz spaces of functions, adelic Poisson summation

Remark: It turns out that the *ad hoc* classsical manipulations of *congruence conditions* (strangely, continuing to this day) is rendered transparent and sensible by re-packaging them as p-adic and adelic Fourier transforms.

This organizational principle applies not only to zeta functions and $GL(1)$ L-functions, but also to automorphic forms for $GL(2)$ and $GL(n)$ and other groups.