Harmonic analysis, on $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$, $\mathbb{Q}_p$, $\mathbb{A}$, and $\mathbb{A}_k/k$, key ingredients in Iwasawa-Tate:

Characters and *Fourier transforms* on $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{A}$, and for completions and adeles of number fields.

Fourier inversion expresses nice functions as *superpositions* (integrals) of characters.

Schwartz spaces $\mathscr{S}(\mathbb{R})$, $\mathscr{S}(\mathbb{Q}_p)$, $\mathscr{S}(\mathbb{A})$ of very-nice functions on $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{A}$, and k_v, $\mathbb{A}_k$ for number fields k.

Adelic Poisson summation

$$\sum_{x\in k} f(x) \;=\; \sum_{x\in k} \widehat{f}(x) \qquad\qquad (\text{for } f \in \mathscr{S}\mathbb{A}_k)$$

Recap:

Theorem: $\mathbb{Q}_p^\vee \approx \mathbb{Q}_p$ and $\mathbb{A}^\vee \approx \mathbb{A}$. Similarly, for number fields k, $k_v^\vee \approx k_v$ and $\mathbb{A}_k^\vee \approx \mathbb{A}_k$.

Corollary: Given *non-trivial* $\psi \in \mathbb{Q}_p^\vee$, every other element of $\mathbb{Q}_p^\vee$ is of the form $x \to \psi(\xi \cdot x)$ for some $\xi \in \mathbb{Q}_p$. Similarly, given *non-trivial* $\psi \in \mathbb{A}^\vee$, every other element of $\mathbb{A}^\vee$ is of the form $x \to \psi(\xi \cdot x)$ for some $\xi \in \mathbb{A}$, ... and analogously for number fields k, ...

Remark: This sort of result is already familiar from the analogue for $\mathbb{R}$, that $x \to e^{i\xi x}$ for $\xi \in \mathbb{R}$ are all the unitary characters of $\mathbb{R}$.

The **standard character** ψ_1 on $\mathbb{Q}_p$ is as follows: given $x \in \mathbb{Q}_p$, there is $x' \in p^{-k}\mathbb{Z}$ for some $k \in \mathbb{Z}$, such that $x - x' \in \mathbb{Z}_p$, and

$$\psi_1(x) = e^{-2\pi i x'} \qquad \text{(sign choice for later)}$$

For $\xi \in \mathbb{Q}_p$, let

$$\psi_\xi(x) \;=\; \psi_1(\xi \cdot x) \qquad\qquad (\text{for } x, \xi \in \mathbb{Q}_p)$$

For a finite extension k_v of $\mathbb{Q}_p$, the **standard character** is

$$\psi_\xi(x) \;=\; \psi_1\Big(\mathrm{tr}^{k_v}_{\mathbb{Q}_p}(\xi \cdot x)\Big) \qquad\qquad (\text{for } x, \xi \in k_v)$$

Since $\mathrm{tr}(\mathfrak{o}_v) \subset \mathbb{Z}_p$, certainly $\ker \psi_\xi \supset \xi^{-1}\mathfrak{o}_v$.

Compact-discrete duality For abelian G, pointwise multiplication makes $\widehat{G}$ an abelian group. The reasonable topology on $\widehat{G}$ is the *compact-open* topology.

Proposition (later): the compact-open topology makes $\widehat{G}$ an abelian (locally-compact, Hausdorf) topological group.

Theorem: The unitary dual of a *compact* abelian group is *discrete*. The unitary dual of a *discrete* abelian group is *compact*.

Theorem: $(\mathbb{A}/k)\widehat{\ } \approx k$. In particular, given any non-trivial character ψ on $\mathbb{A}/k$, *all* characters on $\mathbb{A}/k$ are of the form $x \to \psi(\alpha \cdot x)$ for some $\alpha \in k$.

Proof: (again...) For a (discretely topologized) number field k with adeles $\mathbb{A}$, $\mathbb{A}/k$ is *compact*, and $\mathbb{A}$ is *self-dual*.

Because $\mathbb{A}/k$ is compact, $(\mathbb{A}/k)\widehat{\ }$ is *discrete*. Since multiplication by elements of k respects cosets $x + k$ in $\mathbb{A}/k$, the unitary dual has a k-vectorspace structure given by

$$(\alpha \cdot \psi)(x) = \psi(\alpha \cdot x) \qquad \text{(for } \alpha \in k,\ x \in \mathbb{A}/k\text{)}$$

The quotient map $\mathbb{A} \to \mathbb{A}/k$ gives a natural *injection* $(\mathbb{A}/k)\widehat{\ } \to \widehat{\mathbb{A}}$.

Given non-trivial $\psi \in (\mathbb{A}/k)\widehat{\ }$, the k-vectorspace $k \cdot \psi$ inside $(\mathbb{A}/k)\widehat{\ }$ injects to a copy of $k \cdot \psi$ inside $\widehat{\mathbb{A}} \approx \mathbb{A}$. *Assuming* that the image in $\mathbb{A}$ is essentially the same as the diagonal copy of k, $(\mathbb{A}/k)\widehat{\ }/k$ injects to $\mathbb{A}/k$. The topology of $(\mathbb{A}/k)\widehat{\ }$ is discrete, and the quotient $(\mathbb{A}/k)\widehat{\ }/k$ is still discrete. These maps are continuous group homs, so the image of $(\mathbb{A}/k)\widehat{\ }/k$ in $\mathbb{A}/k$ is a discrete subgroup of a compact group, so is *finite*.

Since $(\mathbb{A}/k)\widehat{\ }$ is a k-vectorspace, $(\mathbb{A}/k)\widehat{\ }/k$ finite implies it is a *singleton*. Thus, $(\mathbb{A}/k)\widehat{\ } \approx k$, assuming the image of $k \cdot \psi$ in $\mathbb{A} \approx \widehat{\mathbb{A}}$ is the usual diagonal copy.

To see how $k \cdot \psi$ is imbedded in $\mathbb{A} \approx \widehat{\mathbb{A}}$, fix non-trivial ψ on $\mathbb{A}/k$, and let ψ be the corresponding character on $\mathbb{A}$. Self-duality of $\mathbb{A}$ asserts $x \to (y \to \psi(xy))$ gives an *isomorphism* $k \to (\mathbb{A}/k)^\vee$. The subgroup $x \cdot \psi$ with $x \in k$ is indeed the usual diagonal copy. ///

Next: Fourier transforms, Fourier inversion, Schwartz spaces of functions, adelic Poisson summation.

Remark: *ad hoc* classical manipulations of *congruence conditions* (strangely, continuing to this day) are rendered transparent and sensible when re-packaged as p-adic and adelic Fourier transforms.

Unsurprisingly, the Fourier transform on k_v is

$$\mathscr{F} f(\xi) \;=\; \widehat{f}(\xi) \;=\; \int_{k_v} \overline{\psi}_\xi(x)\, f(x)\, dx$$

where, given the characters ψ_ξ (for example, the standard ones), the Haar measures are normalized so that *Fourier inversion* holds exactly:

$$f(x) \;=\; \int_{k_v} \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi \qquad\qquad \text{(for nice functions } f\text{)}$$

It is not obvious that Fourier inversion could hold at all...

Recall how/why Fourier inversion works on $\mathbb{R}$. First, a natural approach fails, but suggestively:

$$\int_{\mathbb{R}} \psi_\xi(x)\, \widehat{f}(\xi)\; d\xi \;=\; \int_{\mathbb{R}} \psi_\xi(x) \Big(\int_{\mathbb{R}} \overline{\psi}_\xi(t)\, f(t)\, dt \Big)\; d\xi$$

$$= \int_{\mathbb{R}} f(t) \Big(\int_{\mathbb{R}} \psi_\xi(x-t)\, dt \Big)\; dt$$

If we could *justify* asserting that the inner integral is $\delta_x(t)$, which it *is*, then Fourier inversion follows:

$$\int_{\mathbb{R}} \psi_\xi(x)\, \widehat{f}(\xi)\; d\xi \;=\; \int_{\mathbb{R}} f(t)\, \delta_x(t)\; dt \;=\; f(x)$$

However, this is circular: Fourier inversion, and more, is used to make sense of that inner integral in the first place.

The usual space $\mathscr{S}(\mathbb{R})$ of *Schwartz functions* on $\mathbb{R}$ consists of infinitely-differentiable functions all of whose derivatives are of *rapid decay*, decaying more rapidly at $\pm\infty$ than every $1/|x|^N$. Its topology is given by semi-norms

$$\nu_{k,N}(f) \;=\; \sup_{0\le i\le k}\,\sup_{x\in\mathbb{R}} \Big((1+|x|)^N \cdot \big|f^{(i)}(x)\big|\Big)$$

for $0 \le k \in \mathbb{Z}$ and $0 \le N \in \mathbb{Z}$.

There are countably-many associated (pseudo-) metrics $d_{k,N}(f,g) = \nu_{k,N}(f-g)$, so $\mathscr{S}(\mathbb{R})$ is naturally *metrizable*.

The usual two-or-three-epsilon arguments show that $\mathscr{S}(\mathbb{R})$ is *complete* metrizable.

Remark: There is *no* canonical metric on $\mathscr{S}(\mathbb{R})$, despite the space being unambiguously *metrizable*.

When we know how to justify moving the differentiation under the integral,

$$\frac{d}{d\xi}\widehat{f}(\xi) \;=\; \frac{d}{d\xi}\int_{\mathbb{R}} \overline{\psi}_\xi(x)\, f(x)\, dx \;=\; \int_{\mathbb{R}} \frac{\partial}{\partial x}\overline{\psi}_\xi(x)\, f(x)\, dx$$

$$= \int_{\mathbb{R}} (-2\pi i x)\, \overline{\psi}_\xi(x)\, f(x)\, dx \;=\; (-2\pi i x)\widehat{f}(\xi)$$

Similarly, with an integration by parts,

$$-2\pi i\xi \cdot \widehat{f}(\xi) \;=\; \int_{\mathbb{R}} \frac{\partial}{\partial x}\overline{\psi}_\xi(x)\cdot f(x)\, dx \;=\; -\mathscr{F}\frac{df}{dx}(\xi)$$

It follows that $\mathscr{F}$ maps $\mathscr{S}(\mathbb{R})$ to itself, and, further, is an isomorphism of topological vector spaces.

Despite the impasse in the natural argument for Fourier inversion, the situation is encouraging. A dummy *convergence factor* will legitimize the idea.

For example, let $g(x) = e^{-\pi x^2}$ be the Gaussian. It is its own Fourier transform: moving the contour of integration after a change of variables,

$$\int_{\mathbb{R}} e^{-2\pi i\xi x}\, e^{-\pi x^2}\, dx \;=\; \int_{\mathbb{R}} e^{-\pi(x+i\xi)^2}\, e^{-\pi\xi^2}\, dx$$

$$= \; e^{-\pi\xi^2} \int_{i\xi+\mathbb{R}} e^{-\pi x^2}\, dx \;=\; e^{-\pi\xi^2} \int_{\mathbb{R}} e^{-\pi x^2}\, dx \;=\; e^{-\pi\xi^2}$$

For $f \in \mathscr{S}(\mathbb{R})$, with $f_\varepsilon(x) = f(\varepsilon \cdot x)$, as $\varepsilon \to 0^+$ the dilated f_ε approaches $f(0)$ uniformly on compacts. Thus, as $\varepsilon \to 0^+$, the dilated Gaussian $g_\varepsilon(x) = g(\varepsilon \cdot x)$ approaches 1 uniformly on compacts.

Thus,

$$\int_{\mathbb{R}} \psi_\xi(x)\,\widehat{f}(\xi)\,d\xi \;=\; \int_{\mathbb{R}} \lim_{\varepsilon\to 0^+} g(\varepsilon\xi)\,\psi_\xi(x)\,\widehat{f}(\xi)\,d\xi$$

$$= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} g(\varepsilon\xi)\,\psi_\xi(x)\,\widehat{f}(\xi)\,d\xi$$

by *monotone convergence* or more elementary reasons.

The iterated integral can be legitimately rearranged:

$$\int_{\mathbb{R}} g(\varepsilon\xi)\,\psi_\xi(x)\,\widehat{f}(\xi)\,d\xi \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} g(\varepsilon\xi)\,\psi_\xi(x)\,\overline{\psi}_\xi(t)\,f(t)\,dt\,d\xi$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} g(\varepsilon\xi)\,\psi_\xi(x-t)\,f(t)\,d\xi\,dt$$

Changing variables in the definition of Fourier transform shows $\widehat{g_\varepsilon} = \frac{1}{\varepsilon} g_{1/\varepsilon}$. Thus,

$$\int_{\mathbb{R}} \psi_\xi(x)\,\widehat{f}(\xi)\,d\xi \;=\; \int_{\mathbb{R}} \frac{1}{\varepsilon}\,g\Big(\frac{x-t}{\varepsilon}\Big)\;f(t)\,dt \;=\; \int_{\mathbb{R}} \frac{1}{\varepsilon}\,g\Big(\frac{t}{\varepsilon}\Big)\cdot f(x+t)\,dt$$

The functions $g_{1/\varepsilon}/\varepsilon$ are not an *approximate identity* in the strictest sense, since the supports do not shrink to $\{0\}$. Nevertheless, the integral of each is 1, and as $\varepsilon \to 0^+$, the mass is concentrates on smaller and smaller neighborhoods of $0 \in \mathbb{R}$.

Thus, for $f \in \mathscr{S}(\mathbb{R})$, we have *Fourier inversion*

$$\int_{\mathbb{R}} \psi_\xi(x)\,\widehat{f}(\xi)\,d\xi \;=\; \ldots \;=\; \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \frac{1}{\varepsilon}\,g\Big(\frac{t}{\varepsilon}\Big)\cdot f(x+t)\,dt \;=\; f(x)$$

An analogous argument succeeds for $\mathbb{Q}_p$, but is actually much simpler... (!)

Before describing the Schwartz space $\mathscr{S}(\mathbb{Q}_p)$ and proving Fourier inversion, sample computations of Fourier transforms are useful.

In particular, we need a simply-described function on $\mathbb{Q}_p$ which is its own Fourier transform, to play a role analogous to that of the Gaussian in the archimedean case.

Claim: With Fourier transform on $\mathbb{Q}_p$ defined via the standard character $\psi_1(x) = e^{-2\pi i x'}$ (where $x' \in p^{-\infty}\mathbb{Z}_p$ and $x - x' \in \mathbb{Z}_p$), the characteristic function of $\mathbb{Z}_p$ is its own Fourier transform.

Proof: Let f be the characteristic function of $\mathbb{Z}_p$. Then

$$\widehat{f}(\xi) \;=\; \int_{\mathbb{Q}_p} \overline{\psi}_\xi(x)\, f(x)\, dx \;=\; \int_{\mathbb{Z}_p} \overline{\psi}_1(\xi\cdot x)\, dx \;=\; \int_{\mathbb{Z}_p} \psi_1(-\xi\cdot x)\, dx$$

Recall a form of the *cancellation lemma:* (a tiny case of *Schur orthogonality...*)

Lemma: Let $\psi : K \to \mathbb{C}^\times$ be a continuous group homomorphism on a compact group K. Then

$$\int_K \psi(x)\, dx \;=\; \begin{cases} \text{meas}\,(K) & (\text{for } \psi = 1) \\ \\ 0 & (\text{for } \psi \neq 1) \end{cases}$$

Proof of Lemma: Yes, of course, he measure is a Haar measure on K. Since K is *compact*, it is *unimodular*.

For ψ trivial, of course the integral is the total measure of K.

For ψ non-trivial, there is $y \in K$ such that $\psi(y) \neq 1$. Using the invariance of the measure, change variables by replacing x by xy:

$$\int_K \psi(x)\, dx = \int_K \psi(xy)\, d(xy) = \int_K \psi(x)\, \psi(y)\, dx = \psi(y) \int_K \psi(x)\, dx$$

Since $\psi(y) \neq 1$, the integral is 0. ///

Apply the lemma to the integrals computing the Fourier transform of the characteristic function f of $\mathbb{Z}_p$. Giving the compact group $\mathbb{Z}_p$ measure 1,

$$\widehat{f}(\xi) \;=\; \int_{\mathbb{Z}_p} \psi_1(-\xi\cdot x)\,dx \;=\; \begin{cases} 1 & (\psi_1(-\xi x) = 1 \text{ for } x \in \mathbb{Z}_p) \\ \\ 0 & (\text{otherwise}) \end{cases}$$

On one hand, for $\xi \in \mathbb{Z}_p$, certainly $\psi_1(\xi x) = 1$ for $x \in \mathbb{Z}_p$. On the other hand, for $\xi \notin \mathbb{Z}_p$, there is $x \in \mathbb{Z}_p$ such that, for example, $\xi \cdot x = 1/p$. Then

$$\psi_1(-\xi\cdot x) \;=\; \psi_1(\tfrac{-1}{p}) \;=\; e^{+2\pi i\cdot \frac{1}{p}} \;\neq\; 1$$

Thus, ψ_ξ is not trivial on $\mathbb{Z}_p$, so the integral is 0. Thus, the characteristic function of $\mathbb{Z}_p$ is its own Fourier transform. ///