Harmonic analysis, on $\mathbb{A}_k/k$, adelic Poisson summation.

Corollary: Given *non-trivial* $\psi \in \mathbb{A}^\vee$, every other element of $\mathbb{A}^\vee$ is of the form $x \to \psi(\xi \cdot x)$ for some $\xi \in \mathbb{A}$.

The **standard character** ψ_1 on $\mathbb{Q}_p$ is as follows: given $x \in \mathbb{Q}_p$, there is $x' \in p^{-k}\mathbb{Z}$ for some $k \in \mathbb{Z}$, such that $x - x' \in \mathbb{Z}_p$, and

$$\psi_1(x) \;=\; e^{-2\pi i x'} \qquad \text{(sign choice for later)}$$

For $\xi \in \mathbb{Q}_p$, let

$$\psi_\xi(x) \;=\; \psi_1(\xi \cdot x) \qquad (\text{for } x, \xi \in \mathbb{Q}_p)$$

For a finite extension k_v of $\mathbb{Q}_p$, the **standard character** is

$$\psi_\xi(x) \;=\; \psi_1\big(\mathrm{tr}^{k_v}_{\mathbb{Q}_p}(\xi \cdot x)\big) \qquad (\text{for } x, \xi \in k_v)$$

Probably use these without further comment.

Fourier transform on archimedean or p-adic k_v is

$$\mathscr{F} f(\xi) \;=\; \widehat{f}(\xi) \;=\; \int_{k_v} \overline{\psi}_\xi(x)\, f(x)\, dx$$

Fourier inversion

$$f(x) \;=\; \int_{k_v} \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi \qquad\qquad \text{(for nice functions } f\text{)}$$

The usual space $\mathscr{S}(\mathbb{R})$ of *Schwartz functions* on $\mathbb{R}$ consists of infinitely-differentiable functions all of whose derivatives are of *rapid decay*, decaying more rapidly at $\pm\infty$ than every $1/|x|^N$. Its topology is given by semi-norms

$$\nu_{k,N}(f) \;=\; \sup_{0\le i\le k}\, \sup_{x\in\mathbb{R}} \Big((1+|x|)^N \cdot \big| f^{(i)}(x) \big| \Big)$$

for $0 \le k \in \mathbb{Z}$ and $0 \le N \in \mathbb{Z}$.

Theorem: $\mathscr{F}$ is a topological isomorphism $\mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$.

p-adic Fourier transforms, inversions:

Claim: The characteristic function of $\mathbb{Z}_p$ is its own Fourier transform. ///

Cancellation Lemma: For continuous group hom $\psi : K \to \mathbb{C}^\times$ on a compact group K,

$$\int_K \psi(x)\, dx \;=\; \begin{cases} \text{meas}\,(K) & (\text{for } \psi = 1) \\ \\ 0 & (\text{for } \psi \neq 1) \end{cases}$$

Claim: Characteristic function of $p^k\mathbb{Z}_p$ is p^{-k} times the characteristic function of $p^{-k}\mathbb{Z}_p$. ///

Claim: Characteristic function of $\mathbb{Z}_p + y$ is ψ_y times the characteristic function of $\mathbb{Z}_p$. ///

Combining the two computations above,

$$\mathscr{F}\Big(\text{char fcn } p^k\mathbb{Z}_p + y\Big) \;=\; \psi_y \cdot p^{-k} \cdot (\text{char fcn } p^{-k}\mathbb{Z}_p)$$

Conveniently, products $\psi_y \cdot (\text{char fcn } p^{-k}\mathbb{Z}_p)$ are in the same class of functions, since ψ_y has a kernel which is an open (and compact) neighborhood of 0, so *Fourier transform sends this class of functions is mapped to itself under Fourier transform.*

Schwartz functions $\mathscr{S}(\mathbb{Q}_p)$ on $\mathbb{Q}_p$ are these *special simple functions*, that is, finite linear combinations of characteristic functions of sets $p^k\mathbb{Z}_p + y$.

p**-adic Fourier inversion**:

$$f(x) \;=\; \int_{\mathbb{Q}_p} \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi \qquad\qquad (\text{for } f \in \mathscr{S}(\mathbb{Q}_p))$$

Thus, $\mathscr{F} : \mathscr{S}(\mathbb{Q}_p) \to \mathscr{S}(\mathbb{Q}_p)$ is a *bijection.*

Earlier, we proved that $\mathscr{S}(\mathbb{Q}_p)$ is *dense* in $C_c^o(\mathbb{Q}_p)$.

Schwartz functions $\mathscr{S}(\mathbb{A})$ on the adeles are finite linear combinations of *monomial* functions

$$\Big(\bigotimes_{v\le\infty} f_v\Big)(\{x_v\}) \;=\; \prod_v f_v(x_v)$$

with $f_v \in \mathscr{S}(\mathbb{Q}_v)$, and where *for all but finitely-many* v the local function f_v is the characteristic function of $\mathbb{Z}_v$.

Fourier transform on $\mathscr{S}(\mathbb{A})$ is the product of all the local Fourier transforms, and Fourier inversion follows for $\mathscr{S}(\mathbb{A})$ because it holds for each $\mathscr{S}(\mathbb{Q}_v)$.

Identical definitions and properties apply to all number fields k, their completions k_v, and adeles $\mathbb{A} = \mathbb{A}_k$, with nearly identical proofs.

The harmonic analysis on $\mathbb{R}$ really is parallel to that on $\mathbb{Q}_p$ and $\mathbb{A}$ in many regards. For example,

Plancherel theorem: As on $\mathbb{R}$, $\int_{\mathbb{Q}_p} \widehat{f} \cdot \overline{\widehat{g}} = \int_{\mathbb{Q}_p} f \cdot \overline{g}$ for $f, g \in \mathscr{S}(\mathbb{Q}_p)$.

Proof: The key point is the surjectivity of $\mathscr{F} : \mathscr{S}(\mathbb{Q}_p) \to \mathscr{S}(\mathbb{Q}_p)$:

$$\int_{\mathbb{Q}_p} f \cdot \overline{g} = \int_{\mathbb{Q}_p} f \cdot \overline{\mathscr{F}^{-1}\widehat{g}} = \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} f(x) \cdot \psi_1(-\xi x) \cdot \overline{\widehat{g}}(\xi)\, d\xi\, dx$$

$$= \int_{\mathbb{Q}_p} \Big(\int_{\mathbb{Q}_p} f(x) \cdot \psi_1(-\xi x)\, dx \Big) \cdot \overline{\widehat{g}}(\xi)\, d\xi = \int_{\mathbb{Q}_p} \widehat{f} \cdot \overline{\widehat{g}}$$

This is the same proof as for $\mathbb{R}$, and also applies to $\mathbb{A}$. ///

Then $\mathscr{F}$ is extended to $L^2(\mathbb{Q}_p)$ *by continuity*, giving the *Fourier-Plancherel* transform, no longer defined literally by the integrals.

Fourier series on $\mathbb{A}/k$: For a unimodular topological group G, let $L^2(G)$ be the *completion* of $C_c^o(G)$ with respect to the usual L^2-norm given by

$$|f|^2 = \int_G |f(g)|^2 \, dg \qquad \text{(for } f \in C_c^o(G))$$

Remark: The measurable-function version of $L^2(G)$ *contains* this completion, and is provably equal, but we only need integrals of continuous compactly-supported functions.

The usual *inner product* is

$$\langle f, F \rangle = \int_G f \cdot \overline{F}$$

As usual, the completeness makes $L^2(G)$ a *Hilbert space.*

Remark: Defining or characterizing $L^2(G)$ as the completion of $C_C^o(G)$ makes it complete. In contrast, giving $L^2(G)$ as the collection of *measurable functions* meeting a condition leaves us needing to *prove* completeness.

(big) **Theorem:** For a *compact abelian* group G, with total measure 1, the continuous group homomorphisms (*characters*) $\psi : G \to \mathbb{C}^\times$ form an orthonormal *Hilbert-space basis* for $L^2(G)$. That is,

$$L^2(G) \;=\; \text{completion of} \bigoplus_{\psi \in G^\vee} \mathbb{C} \cdot \psi$$

and

$$f \;=\; \sum_{\psi \in G^\vee} \langle f, \psi \rangle \cdot \psi \qquad \text{(for } f \in L^2(G)\text{, convergence in } L^2(G)\text{)}$$

Remark: This applies to the circle $\mathbb{R}/\mathbb{Z}$!

Remark: Recall that a *Hilbert-space* basis of a Hilbert space V is not a *vector-space* basis for V, but for a dense subspace.

Remark: For *finite* abelian groups, this follows from the spectral theorem for commuting *unitary* operators on finite-dimensional $\mathbb{C}$-vectorspaces. (See 2010-11 notes.)

Remark: As in the elementary example of the circle $\mathbb{R}/\mathbb{Z}$, convergence in L^2 says nothing directly about *pointwise* convergence, much less *uniform* pointwise convergence.

Proof: Orthonormality is easy: for $\psi \neq \varphi$ characters,

$$\langle \psi, \varphi \rangle \;=\; \int_G \psi(g) \cdot \overline{\varphi}(g)\; dg \;=\; \int_G \psi\varphi^{-1}(g)\; dg$$

By the *cancellation lemma*, this is 0 for $\psi \neq \varphi$.

Completeness is more serious. We must prove existence of sufficiently many *eigenvectors* for the action of G on complex-valued functions

$$g \cdot f(x) \;=\; f(xg) \qquad\qquad (\text{for } f \in C^o_c(G) \text{ and } x, g \in G)$$

For f to be an *eigenfunction* means that

$$g \cdot f \;=\; \lambda_f(g) \cdot f \qquad\qquad (\text{for all } g \in G, \text{ with } \lambda_f(g) \in \mathbb{C})$$

The *unitariness* is

$$\langle g \cdot f,\, g \cdot F \rangle \;=\; \int_G f(xg)\, \overline{\varphi}(xg)\, dx \;=\; \int_G f(x)\, \overline{\varphi}(x)\, dx \;=\; \langle f, F \rangle$$

The eigenvalues $\lambda_f(g)$ cannot be unrelated: for $g, h \in G$,

$$\lambda_f(gh) \cdot f \;=\; (gh) \cdot f \;=\; g \cdot (h \cdot f) \;=\; g \cdot (\lambda_f(h)\, f)$$

$$=\; \lambda_f(h)\; g \cdot f \;=\; \lambda_f(h)\, \lambda_f(g)\, f$$

so $\lambda_f : G \to \mathbb{C}^\times$ is a *group homomorphism.*

For G *finite*, $L^2(G)$ is finite-dimensional. By finite-dimensional spectral theory for *unitary* operators, $L^2(G)$ is a direct sum of eigenspaces V_λ, for group homomorphism $\lambda : G \to \mathbb{C}^\times$.

Each eigenfunction f is itself a constant multiple of a group homomorphism $G \to \mathbb{C}^\times$:

$$f(x) \;=\; f(1 \cdot x) \;=\; \lambda_f(x)\, f(1)$$

If $\lambda_f = \lambda_F$, with normalization $f(1) = 1 = F(1)$,

$$f(x) \;=\; f(1 \cdot x) \;=\; \lambda_f(x)\, f(1) \;=\; \lambda_F(x)\, F(1) \;=\; F(x)$$

That is, each λ_f occurs with *multiplicity one.*

Certainly every group homomorphism $G \to \mathbb{C}^\times$ is a complex-valued function on finite G, so

$$L^2(G) \;=\; \bigoplus_{\psi \in G^\vee} \mathbb{C} \cdot \psi \qquad\qquad (G \text{ finite abelian})$$

We did *not* use the structure theorem for finite abelian groups.

On infinite-dimensional Hilbert spaces, even for *unitary* operators, general spectral theory does *not* guarantee *eigenvectors.*

From a spectral viewpoint, the best operators on infinite-dimensional Hilbert spaces are *self-adjoint compact* operators.

The *self-adjointness* is the usual $\langle Tv, w\rangle = \langle v, Tw\rangle$.

The *compactness* is that the image TB of the unit ball B has *compact closure.* Thus, the image $\{Tv_i\}$ of a *bounded* sequence $\{v_i\}$ has a *convergent subsequence* $\{T_{v_{i_k}}\}$.

On finite-dimensional vector spaces, *every* linear operator is compact.

One of the most useful theorems in the universe:

Theorem: Let R be a set of compact, self-adjoint, mutually commuting operators on a Hilbert space V. Suppose the action is *non-degenerate* in the sense that for $0 \neq v \in V$ there is $T \in R$ with $Tv \neq 0$. Then V has an *orthonormal* Hilbert-space basis of *simultaneous eigenvectors* for R. The joint eigenspaces are finite-dimensional.

[The simple proof is below. Other useful details arise.]

Where do compact operators come from?

From *integral operators*, sometimes misleadingly called *convolution operators*. This misnomer is understandable, but does make less intelligible what's going on.

A function $\eta \in C_c^o(G)$ acts on $L^2(G)$ by the integral operator

$$(\eta \cdot f)(x) \;=\; \int_G \eta(g)\, f(xg)\, dg$$

There is the compatibility

$$\alpha \cdot (\beta \cdot f)(x) \;=\; \int_G \int_G \alpha(h)\, \beta(g)\, f(xhg)\; dg\, dh$$

$$=\; \int_G \Big(\int_G \alpha(hg^{-1})\, \beta(g)\, dg \Big) f(xh)\; dh$$

$$=\; \int_G (\alpha * \beta)(h)\; f(xh)\; dh \;=\; \big((\alpha * \beta) \cdot f\big)(x)$$

That $\alpha * \beta$ *is* convolution, indeed, but the action on a vector space on which G acts is much more general than convolution. Further, there is a discrepancy of inverse-or-not if we try to force the action of $C_c^o(G)$ on $L^2(G)$ to be convolution.

An innocent change of variables gives

$$(\alpha \cdot f)(x) \;=\; \int_G \alpha(y)\, f(xy)\; dy \;=\; \int_G \alpha(x^{-1}y)\; f(y)\; dy$$

Write $K(x,y) = \alpha(x^{-1}y)$ to suggest viewing $\alpha(x^{-1}y)$ as a *kernel* for an *integral operator*, analogous to a *matrix*, but indexed by $x, y \in G$.

Claim: For topological spaces X, Y with nice measures, for $K(x,y) \in C_c^o(X \times Y)$, the linear operator $T : L^2(Y) \to L^2(X)$ by

$$Tf(x) \;=\; \int_Y K(x,y)\, f(y)\, dy$$

is *compact*. For $X = Y$ and $K(y,x) = \overline{K(x,y)}$, the operator T is *self-adjoint*.

Remark: Fredholm, Volterra, Hilbert, Riesz, and others inverted certain ordinary *differential* operators *(Sturm-Liouville problems)* to *integral* operators, which happened to be *compact*, thus giving a basis of eigenfunctions, enabling solution of such problems.

Remark This same strategy applies to compact G that are not necessarily *abelian*, to decompose $L^2(G)$ into *irreducible representations*, although most of the irreducibles are not one-dimensional, *not* spanned by group homomorphisms $G \to \mathbb{C}^\times$. Even for G *non-compact, non-abelian*, for discrete subgroups Γ with $\Gamma\backslash G$ *compact*, the same mechanism decomposes $L^2(\Gamma\backslash G)$.

Proof of spectral theorem for compact self-adjoint operators: The key point of the theorem above is the spectral theorem for a single self-adjoint compact operator $T : V \to V$.

Lemma: A continuous *self-adjoint* operator T on a Hilbert space V has operator norm $|T| = \sup_{|v|\leq 1} |Tv|$ expressible as

$$|T| \;=\; \sup_{|v|\leq 1} |\langle Tv, v\rangle|$$

Proof of Lemma: On one hand, certainly $|\langle Tv, v\rangle| \;\leq\; |Tv| \cdot |v|$, giving the easy direction of inequality.

On the other hand, let $\sigma \;=\; \sup_{|v|\leq 1} |\langle Tv, v\rangle|$. A polarization identity gives

$$2\langle Tv, w\rangle + 2\langle Tw, v\rangle \;=\; \langle T(v+w), v+w\rangle - \langle T(v-w), v-w\rangle$$

With $w \;=\; t \cdot Tv$ with $t \;>\; 0$, since $T \;=\; T^*$, both $\langle Tv, w\rangle$ and $\langle Tw, v\rangle$ are non-negative real. Taking absolute values,

$$
\begin{aligned}
4\langle Tv, t\cdot Tv\rangle \;&=\; \sigma\cdot|v+t\cdot Tv|^2+\sigma\cdot|v-t\cdot Tv|^2 \\
=\; &\Big|\langle T(v+t\cdot Tv), v+t\cdot Tv\rangle-\langle T(v-t\cdot Tv), v-t\cdot Tv\rangle\Big| \\
\le\; &\sigma\cdot|v+t\cdot Tv|^2+\sigma\cdot|v-t\cdot Tv|^2 \;=\; 4\sigma\cdot\big(|v|^2+t^2\cdot|Tv|^2\big)
\end{aligned}
$$

Divide through by $4t$ and set $t = |v|/|Tv|$ to minimize the right-hand side, obtaining

$$|Tv|^2 \;\le\; \sigma\cdot|v|\cdot|Tv|$$

giving the other inequality, proving the Lemma. ///

Key Lemma: A compact self-adjoint operator T has largest eigenvalue $\pm|T|$.

Proof of Key Lemma: Take $|T| > 0$, or else $T = 0$. Using the characterization of operator norm, let v_i be a sequence of unit vectors such that $|\langle Tv_i, v_i\rangle| \to |T|$. On one hand, using $\langle Tv, v\rangle = \langle v, Tv\rangle = \overline{\langle Tv, v\rangle}$,

$$0 \le |Tv_i - \lambda v_i|^2 = |Tv_i|^2 - 2\lambda\langle Tv_i, v_i\rangle + \lambda^2|v_i|^2$$

$$\le \lambda^2 - 2\lambda\langle Tv_i, v_i\rangle + \lambda^2$$

By assumption, the right-hand side goes to 0. Using compactness, replace v_i with a subsequence such that Tv_i has limit w. Then the inequality shows that $\lambda v_i \to w$, so $v_i \to \lambda^{-1}w$. Thus, by continuity of T, $Tw = \lambda w$. ///

The commutativity of the set R of operators ensures that the operators stabilize each others' eigenspaces. The non-degeneracy ensures that the orthogonal complement of all the joint eigenspaces is $\{0\}$. ///

Next, prove that $K(x,y) \in C_c^o(X \times Y)$ gives a compact operator...