Iwasawa-Tate on ζ-functions and L-functions

After the main part, namely, *analytic continuation* and *functional equation* of global zeta integrals...

- Archimedean Fourier transforms: Hecke's identity
- Convergence of global half-zeta integrals
- Proof of Hecke's identity

Archimedean Fourier transform: Hecke's identity

Recall:

On $\mathbb{R}$, the Gaussian $e^{-\pi x^2}$ is its own Fourier transform.

On $\mathbb{R}$, $xe^{-\pi x^2}$ is multiplied by $-i$ under Fourier transform.

Less obviously: $(x \pm iy)^\ell \, e^{-\pi(x^2+y^2)}$ is an eigenfunction for Fourier transform, with eigenvalue $i^{-\ell}$.

Proof (recap): Do $(x + iy)^\ell \, e^{-\pi(x^2+y^2)}$. Rewrite as

$$\int_{\mathbb{C}} e^{-\pi i(z\overline{w}+\overline{z}w)} \, z^\ell \, e^{-\pi z\overline{z}} \, dz$$

$$= \; (-\pi i)^{-\ell} \Big(\frac{\partial}{\partial \overline{w}}\Big)^\ell \int_{\mathbb{C}} e^{-\pi i(z\overline{w}+\overline{z}w)} \, e^{-\pi z\overline{z}} \, dz \; = \; (-\pi i)^{-\ell} \Big(\frac{\partial}{\partial \overline{w}}\Big)^\ell e^{-\pi w\overline{w}}$$

$$= \; (-\pi i)^{-\ell} (-\pi w)^\ell \, e^{-\pi w\overline{w}} \; = \; i^{-\ell} \cdot w^\ell \, e^{-\pi w\overline{w}}$$

This presumes $\partial/\partial\overline{w}$ works as expected, which it does. ///

Hecke's identity: Let P be a homogeneous, degree d *harmonic* polynomial on $\mathbb{R}^n$, meaning that $\Delta P = 0$, where $\Delta = \sum_j \partial^2/\partial x_j^2$ is the usual Laplacian. Let $\langle x, \xi \rangle = \sum_j x_j \xi_j$ be the usual pairing. Then $P(x)\, e^{-\pi|x|^2}$ is a Fourier transform eigenfunction with eigenvalue i^{-d}:

$$\Big(P(x)\, e^{-\pi|x|^2}\Big)^{\widehat{\ }}(\xi) \;=\; i^{-d} \cdot P(\xi)\, e^{-\pi|\xi|^2}$$

Proof postponed...

Remark: The proof of Hecke's identity will illustrate the nearly-magical strength of *representation theory,* as manifest in eigenfunction problems for invariant differential operators.

Specifically, we will see that harmonic polynomials on $\mathbb{R}^n$ have a useful interpretation as *eigenfunctions* for a rotation-invariant *Laplacian* on the sphere $S^{n-1} \subset \mathbb{R}^n$. Hecke's identity will result from comparison of eigenvalues and multiplicities. (!)

Convergence of half-zeta integrals The point is to genuinely prove convergence of the half-zeta integrals

$$\int_{\mathbb{J}^+} |y|^s\, f(y)\, dy$$

with f a Schwartz function on the adeles, for *all* $s \in \mathbb{C}$, *not* by dis-assembling this and trying to reduce to the classical situation.

For f Schwartz, for all N

$$|f(x)| \ll_{N,f} \prod_v \sup(|x_v|_v, 1)^{-2N} \qquad\qquad (\text{adele } x = \{x_v\})$$

Define the **gauge** on *ideles* y by

$$\nu(y) = \prod_v \sup\{|y_v|_v, \Big|\frac{1}{y_v}\Big|_v\}$$

Almost all factors on the right-hand side are 1, so there is no issue of convergence.

Further, note that

$$\big(\sup\{a,\, 1\}\big)^2 \;=\; \sup\{a^2,\, 1\} \;=\; a \cdot \sup\{a,\, \frac{1}{a}\} \qquad\qquad (\text{for } a > 0)$$

Applying the latter equality to every factor,

$$\prod_v \sup(|y_v|_v, 1)^{-2N} = |y|^{-N} \prod_v \sup(|y_v|_v, \frac{1}{|y_v|_v})^{-N} = |y|^{-N} \nu(y)^{-N}$$

Thus, on $\mathbb{J}^+ = \{|y| \geq 1\}$, with $N \geq 0$,

$$\prod_v \sup(|y_v|_v, 1)^{-2N} \;=\; |y|^{-N} \nu(y)^{-N} \;\leq\; \nu(y)^{-N}$$

Thus, with $\sigma = \mathrm{Re}\, s$, for every $N \geq 0$

$$\Big| \int_{\mathbb{J}^+} |y|^s \, f(y) \, dy \Big| \ll_{f,N} \int_{\mathbb{J}^+} |y|^\sigma \, \nu(y)^{-N} \, dy$$

$$\leq \int_{\mathbb{J}} |y|^\sigma \, \nu(y)^{-N} \, dy \; = \; \prod_v \Big(\int_{k_v^\times} |y|^\sigma \; \sup(|y|, \frac{1}{|y|})^{-N} \, dy \Big)$$

For $N > |\sigma|$, the non-archimedean local integrals are absolutely convergent:

$$\int_{k_v^\times} |y|^\sigma \; \sup(|y|, \frac{1}{|y|})^{-N} \, dy \; = \; \sum_{\ell=0}^{\infty} q_v^{-\sigma-N} \; + \; \sum_{\ell=1}^{\infty} q_v^{\sigma-N}$$

$$= \; \frac{1}{1-q^{-\sigma-N}} + \frac{q^{\sigma-N}}{1-q^{\sigma-N}} \; = \; \frac{1-q^{-2N}}{(1-q^{-\sigma-N})\,(1-q^{\sigma-N})}$$

Note the exponents $2N$, $N+\sigma$, and $N-\sigma$.

The archimedean integrals are convergent for similarly overwhelming reasons.

For $N > \frac{1}{2}$ and $N > |\sigma| + 1$, the product over places is dominated by the Euler product for the completed zeta functions $\xi_k(N+\sigma)\xi_k(N-\sigma)/\xi_k(2N)$, which converges absolutely.

Thus, for all $s \in \mathbb{C}$, for all Schwartz f, the half-zeta integrals

$$\int_{\mathbb{J}^+} |y|^s \, f(y) \, dy = \int_{\mathbb{J}^+/k^\times} |y|^s \, \theta_f^*(y) \, dy \qquad \Big(\text{with } \theta_f^*(y) = \sum_{\alpha \in k^\times} f(\alpha y)\Big)$$

are absolutely convergent. Similarly, for $|\chi| = 1$, the same estimate gives absolute convergence of

$$\int_{\mathbb{J}^+} |y|^s \, \chi(y) \, f(y) \, dy \; = \; \int_{\mathbb{J}^+/k^\times} |y|^s \, \chi(y) \, \theta_f^*(y) \, dy$$

Remark: It would be misguided to try to convert this to a more classical-sounding argument.

Lemma: For all N, a Schwartz function f on $\mathbb{A}$ satisfies

$$|f(x)| \ll_{f,N} \prod_v \sup(|x_v|_v, 1)^{-2N} \qquad \text{(for } x \in \mathbb{A}\text{)}$$

Proof: By definition, $f \in \mathscr{S}(\mathbb{A})$ is a finite sum of *monomials* $f = \bigotimes_v f_v$. Thus, it suffices to consider monomial f, and to prove the *local* assertion that for $f_v \in \mathscr{S}(k_v)$

$$|f_v(x)| \ll_{N,f_v} \sup(|x_v|_v, 1)^{-2N} \qquad \text{(for } x \in k_v\text{)}$$

At archimedean places, the definition of the Schwartz space requires that

$$\sup_{x \in k_v} (1 + |x|_v)^N \cdot |f_v(x)| < \infty \qquad \text{(for archimedean } k_v\text{, for all } N\text{)}$$

Thus, for archimedean k_v,

$$|f_v(x)| \ll_{f,N} (1+|x|_v)^{-2N} \le \sup(|x|_v, 1)^{-2N}$$

Almost everywhere, f_v is the characteristic function of the local integers. At such places,

$$|f_v(x)| = \left\{ \begin{matrix} 1 & (\text{for } |x|_v \le 1) \\ \\ 0 & (\text{for } |x|_v > 1) \end{matrix} \right\} \le \sup(|x_v|_v, 1)^{-2N} \qquad (\text{for all } N)$$

At the remaining *bad* finite primes, $f_v \in \mathscr{S}(k_v)$ is compactly supported and locally compact. Then, similar to the *good* prime case,

$$|f_v(x)| \ll_{f_v} \left\{ \begin{matrix} 1 & (x \in \text{spt} f_v) \\ \\ 0 & (x \notin \text{spt} f_v) \end{matrix} \right\} \ll_{f_v,N} \sup(|x_v|_v, 1)^{-2N} \qquad (\text{for all } N)$$

This proves the lemma. ///

Now the proof of

Hecke's identity: Let P be a homogeneous, degree d *harmonic* polynomial on $\mathbb{R}^n$, meaning that $\Delta P = 0$, where $\Delta = \sum_j \partial^2/\partial x_j^2$ is the usual Laplacian. Let $\langle x, \xi\rangle = \sum_j x_j \xi_j$ be the usual pairing. Then $P(x)\, e^{-\pi|x|^2}$ is a Fourier transform eigenfunction with eigenvalue i^{-d}:

$$\Big(P(x)\, e^{-\pi|x|^2}\Big)^\frown(\xi) \;=\; i^{-d}\cdot P(\xi)\, e^{-\pi|\xi|^2}$$

Proof: Whether or not P is harmonic,

$$\Big(P(x)\, e^{-\pi|x|^2}\Big)^\frown(\xi) \;=\; \int_{\mathbb{R}^n} e^{-2\pi i\langle \xi, x\rangle}\; P(x)\, e^{-\pi|x|^2}\; dx$$

$$=\; P\Big(\frac{1}{-2\pi i}\frac{\partial}{\partial \xi}\Big)\int_{\mathbb{R}^n} e^{-2\pi i\langle \xi, x\rangle}\; e^{-\pi|x|^2}\; dx$$

because

$$P\Big(\frac{1}{-2\pi i}\frac{\partial}{\partial \xi_1}, \ldots, \frac{1}{-2\pi i}\frac{\partial}{\partial \xi_n}\Big) e^{-2\pi i\langle \xi, x\rangle} \;=\; P(x)$$

Since the Gaussian is its own Fourier transform,

$$\Big(P(x)\,e^{-\pi|x|^2}\Big)\widehat{\ }(\xi) \;=\; P\Big(\frac{1}{-2\pi i}\frac{\partial}{\partial\xi}\Big)\,e^{-\pi|\xi|^2}$$

whether or not P is harmonic. Certainly

$$P\Big(\frac{1}{-2\pi i}\frac{\partial}{\partial\xi}\Big)\,e^{-\pi|\xi|^2} \;=\; P^{\#}(\xi)\,e^{-\pi|\xi|^2}$$

for a polynomial $P^{\#}$ of total degree at most that of P. Since Fourier transform commutes with the action of $O(n,\mathbb{R})$ on functions,

$$\Big((P\circ g)(x)\,e^{-\pi|x|^2}\Big)\widehat{\ }(\xi) \;=\; \Big(P(gx)\,e^{-\pi|gx|^2}\Big)\widehat{\ }(\xi)$$

$$=\; \Big(P(x)\,e^{-\pi|x|^2}\Big)\widehat{\ }(g\xi) \;=\; P^{\#}(g\xi)\,e^{-\pi|\xi|^2}$$

Thus, $P \to P^{\#}$ is an $O(n,\mathbb{R})$-map:

$$(P\circ g)^{\#} \;=\; P^{\#}\circ g \qquad\qquad (\text{for } g\in O(n,\mathbb{R}))$$

Thus, $P \to P^{\#}$ gives an $O(n, \mathbb{R})$-respecting map of the space V_d, of *all* polynomials of total degree at most d, to itself.

The sequel: we will show... first, the space H_d of homogeneous degree-d harmonic polynomials is *irreducible* as $O(n, \mathbb{R})$-representation, meaning that it has no proper vector subspace stable under $O(n, \mathbb{R})$.

Second, as $O(n, \mathbb{R})$-representation space, meaning as complex vector space with linear action of $O(n, \mathbb{R})$,

$$V_d \;=\; H_d \;\oplus\; \bigoplus (\text{other irreducibles } \pi \not\approx H_d)$$

Third, *any* $O(n, \mathbb{R})$-respecting map $V_d \to V_d$ maps H_d to itself.

Fourth, (an instance of *Schur's Lemma*) that any $O(n, \mathbb{R})$-map of *any* irreducible to itself is a *scalar*.

Fifth, the two-variable case determines the constant i^{-d}.