• **Interlude:** Calculus on spheres: invariant integrals, invariant $\Delta = \Delta^S$, integration-by-parts, etc. Decomposition of $L^2(S^{n-1})$ into Δ^S-eigenfunctions:

$$L^2(S^{n-1}) \;=\; \text{completion of } \bigoplus_d H_d\big|_{S^{n-1}}$$

and $H_d\big|_{S^{n-1}}$ is Δ^S-eigenspace with eigenvalue $\lambda_d = -d(d+n-2)$

Hecke's identity: For a homogeneous, degree d *harmonic* polynomial P on $\mathbb{R}^n$, $P(x)e^{-\pi|x|^2}$ is a Fourier transform eigenfunction with eigenvalue i^{-d}:

$$\Big(P(x)\, e^{-\pi|x|^2}\Big)\widehat{\ }(\xi) \;=\; i^{-d}\cdot P(\xi)\, e^{-\pi|\xi|^2}$$

The *idea* of the proof is that the $SO(n,\mathbb{R})$-map

$$\# \,:\, H_d \;\longrightarrow\; \mathbb{C}[x]^{\le d} \;\approx\; H_d \oplus (\text{other irreducibles})$$

must map H_d to the other copy of H_d, and, by Schur's lemma, be a scalar.

Theorem: *(Schur's Lemma)* For a finite-dimensional irreducible representation V of a group G, any G-intertwining $\varphi : V \to V$ of V to itself is *scalar*. ///

Theorem: The space H_d of harmonic homogeneous total-degree d polynomials is an *irreducible* $SO(n, \mathbb{R})$-representation. ///

The proof uses a *multiplicity-one* result, namely,

Lemma: On $\mathbb{R}^n$, for each fixed d, in H_d there is a unique (up to constant multiples) $SO(n-1, \mathbb{R})$-fixed vector $f \in H_d$. ///

Similar to Schur's Lemma:

Proposition: For *non-isomorphic* irreducible G-representations V, W, the only G-hom $\varphi : V \to W$ is the zero map. ///

Proposition: *(Complete Reducibility)* A finite-dimensional *unitary* G-representation V is a finite orthogonal direct sum of irreducible G-subrepresentations. ///

A direct sum of a number of copies of an irreducible π is denoted

$$m \cdot \pi \ = \ \underbrace{\pi \oplus \ldots \oplus \pi}_{m}$$

Corollary: For non-isomorphic irreducibles $\sigma_1, \ldots, \sigma_n$ and non-negative integers $m_1, \ldots, m_n$, $m'_1, \ldots, m'_n$,

$$\mathrm{Hom}_G(\bigoplus_i m_i\sigma_i, \bigoplus_j m'_j\sigma_j) \ \approx \ \bigoplus_i \mathrm{Hom}_G(m_i\sigma_i, m'_i\sigma_i)$$

$$\approx \ \bigoplus_i \big(m'_i\text{-by-}m_i \text{ complex matrices}\big)$$

Remark: This continues the theme that non-isomorphic irreducibles *do not interact.*

Proof: The characterization of direct sums is that maps from a direct sum are direct sums of maps from the individuals. Finite direct sums (coproducts) of vector spaces (or representations) are also products. Thus, for general categorical reasons,

$$\mathrm{Hom}_G\Big(\bigoplus_i m_i\sigma_i,\ \bigoplus_j m'_j\sigma_j\Big) \;\approx\; \bigoplus_{ij}\bigoplus_{k=1}^{m_i}\bigoplus_{\ell=1}^{m'_j}\mathrm{Hom}_G(\sigma_i,\sigma_j)$$

By Schur's lemma and related ideas, $\mathrm{Hom}_G(\sigma_i,\sigma_j) = 0$ unless $i = j$, in which case it is $\mathbb{C}$. Thus, going backward slightly,

$$\mathrm{Hom}_G\Big(\bigoplus_i m_i\sigma_i,\ \bigoplus_j m'_j\sigma_j\Big) \;\approx\; \bigoplus_i \mathrm{Hom}_G(m_i\sigma_i, m'_i\sigma_i)$$

Further, $\mathrm{Hom}_G(\sigma_i, m'_i\sigma_i)$ is exactly maps $\varphi(v) = c_1 v \oplus \ldots \oplus c_{m'_i} v$ with scalars c_k, by Schur. Sticking these together gives m'_i-by-m_i matrices. ///

Remark: The previous *proof*, not the assertion, is objectionable in two ways. First, expressions of representations as sums of irreducibles have *not* been described intrinsically. Second, there *is* something to do to certify that finite coproducts and products of representation spaces really are coproducts and products of the underlying vector spaces.

Isotypes and co-isotypes For irreducible π of G, specify a G-sub V^π of a G-rep'n V by requiring that any map $m \cdot \pi \to V$ *factors (uniquely) through* V^π, that is, $m \cdot \pi \to V^\pi \subset V$, the π-*isotype* of V. Dually, the π-co-isotype V_π of V is the quotient of V such that any map $V \to m \cdot \pi$ factors through V_π: $V \to V_\pi \to m \cdot \pi$.

A priori, existence is unclear, but on categorical grounds these are unique up to unique isomorphism if they exist.

For *unitary* representations, the kernel of the map to the co-isotype has an orthogonal complement, so the co-isotype is naturally isomorphic to a sub-object.

Happily, for *finite-dimensional* irreducibles π of (compact) G, there is a natural *projector* to the π-isotype.

It is not obvious, but history does reasonably lead to the following. For a finite-dimensional irreducible π of a topological group G, the *character* χ_π of G is a function on G defined by

$$\chi_\pi(g) \;=\; \mathrm{trace}\,\pi(g)$$

The *dual* or *contragredient* $\pi^\vee = \mathrm{Hom}_\mathbb{C}(\pi, \mathbb{C})$ has G-action

$$(g \cdot \lambda)(v) \;=\; \lambda(g^{-1}v) \qquad\qquad (\text{for } v \in \pi \text{ and } \lambda \in \pi^\vee)$$

Proposition: For any G-representation V, the map

$$v \;\longrightarrow\; \chi_\pi \cdot v \;=\; \int_G \chi_\pi(g)\, gv\, dg \;\;\in\;\; V$$

is a G-hom *projecting* $V \to V^{(\pi^\vee)}$, its $\pi^\vee$-isotype.

Proof: First, observe the *conjugation-invariance* of χ_π:

$$\chi_\pi(gxg^{-1}) \;=\; \mathrm{tr}\big(\pi(gxg^{-1})\big) \;=\; \mathrm{tr}\big(\pi(g)\,\pi(x)\,\pi(g)^{-1}\big)$$

$$=\; \mathrm{tr}\,\pi(x) \;=\; \chi_\pi(x)$$

by conjugation-invariance of *trace*. From this, χ_π commutes with the action of G on any representation space V:

$$\chi_\pi \cdot (g \cdot v) \;=\; \int_G \chi_\pi(x)\,x(gv)\;dx \;=\; \int_G \chi_\pi(x)\,(xg)v\;dx$$

$$=\; \int_G \chi_\pi(gxg^{-1})\,(gx)v\;dx \;=\; \int_G \chi_\pi(x)\,g(xv)\;dx$$

$$=\; g\int_G \chi_\pi(x)\,xv\;dx \;=\; g\cdot(\chi_\pi\cdot v)$$

Thus, χ_π gives a G-map of any G-representation V to itself. By Schur's Lemma, χ_π is a *scalar* on irreducibles.

Similarly, for any G-map $\varphi : V \to W$ of G-spaces, and for any continuous function f on G, with $f \cdot v = \int_G f(g)\, gv\, dg$,

$$\varphi(f \cdot v) \;=\; \varphi\Big(\int_G f(g)\, gv\, dg \Big) \;=\; \int_G f(g)\, \varphi(gv)\, dg$$

$$=\; \int_G f(g)\, g\varphi(v)\, dg \;=\; f \cdot \varphi(v)$$

In particular, this applies to $f = \chi_\pi$, and shows that the action on representations depends only on the *isomorphism class* of the representation, not on the model or representative.

For irreducibles $\tau \not\approx \sigma^\vee$, claim that χ_σ acts by 0 on τ. Indeed, letting $\{e_i\}$ be an orthonormal basis for σ, for $v, w \in \tau$, with inner product in τ,

$$\langle \chi_\sigma \cdot v, w \rangle \;=\; \int_G \chi_\sigma(g)\, \langle gv, w \rangle \, dg \;=\; \sum_i \int_G \langle ge_i, e_i \rangle \, \langle gv, w \rangle \, dg$$

Up to a complex conjugation, this is an inner product in $L^2(G)$ of *(matrix) coefficient functions*

$$c^\sigma_{x,y}(g) \;=\; \langle gx, y \rangle_\sigma$$

This brings us to the *Schur inner product formulas:* claim

$$\langle c^\sigma_{v,w},\, c^\tau_{x,y} \rangle_{L^2(G)} \;=\; \begin{cases} 0 & (\text{for } \sigma \not\approx \tau) \\ \dfrac{|G|}{\dim \sigma} \cdot \langle v, x \rangle \cdot \overline{\langle w, y \rangle} & (\text{for } \sigma \approx \tau) \end{cases}$$

Proof: Let $S : \sigma \to L^2(G)$ by $Sv = c_{v,w}$, and $T : \tau \to L^2(G)$ by $Tx = c_{x,y}$. Then

$$\langle c^\sigma_{v,w},\, c^\tau_{x,y}\rangle \;=\; \langle Sv,\, Tx\rangle \;=\; \langle v,\, S^*Tx\rangle$$

By Schur's Lemma $S^*T : \tau \to \sigma$ is a scalar $C = C_{w,y}$ for $\sigma \approx \tau$, and 0 for $\sigma \not\approx \tau$. Take $\tau \approx \sigma$. Then

$$\langle c_{v,w},\, c^\tau_{x,y}\rangle_{L^2(G)} \;=\; C_{w,y} \cdot \langle v, x\rangle$$

Similarly, conjugating and changing variables $h \to h^{-1}$,

$$\langle c_{v,w},\, c_{x,y}\rangle_{L^2(G)} \;=\; C_{v,x} \cdot \overline{\langle w, y\rangle}$$

Thus, with $\tau \approx \sigma$, $\langle c_{v,w},\, c_{x,y}\rangle_{L^2(G)} = C \cdot \langle v, x\rangle \cdot \overline{\langle w, y\rangle}$ with C depending only on $\sigma \approx \tau$. To determine C, using Plancherel,

$$C \cdot \dim \sigma \;=\; \sum_i C \cdot \langle e_1, e_1\rangle \langle e_i, e_i\rangle \;=\; \sum_i \int_G c_{e_1,e_i}(g)\overline{c_{e_1,e_i}(g)}\, dg$$

$$= \int_G \sum_i \langle g e_1, e_i\rangle \overline{\langle g e_1, e_i\rangle}\, dg \;=\; \int_G |g e_1|^2\, dg \;=\; \int_G |e_1|^2\, dg \;=\; |G|$$

This gives C and gives the Schur inner product formula. ///

Returning to χ_σ,

$$\langle \chi_\sigma \cdot v, w\rangle_{\sigma^\vee} \;=\; \int_G \chi_\sigma(g)\, \langle gv, w\rangle_{\sigma^\vee}\, dg$$

$$= \sum_i \int_G \langle g e_i, e_i\rangle_\sigma\, \overline{\langle g^{-1} w, v\rangle}_\sigma\, dg \;=\; \sum_i \langle c_{e_i,e_i},\, c_{w,v}\rangle$$

$$= \frac{|G|}{\dim \sigma} \sum_i \langle e_i, w\rangle \overline{\langle e_i, v\rangle} \;=\; \frac{|G|}{\dim \sigma} \langle v, w\rangle$$

That is, χ_σ is 0 on τ for $\tau \not\approx \sigma^\vee$, and is $|G|/\dim\sigma$ on $\sigma^\vee$.

That is, adjusted by a constant, given *complete reducibility*, $\chi_{\sigma^\vee}$ is a projector to the σ-isotype of a representation of G. ///

Corollary: *(Reprise)* For irreducibles $\sigma \not\approx \tau$, $\mathrm{Hom}_G(\sigma, m\cdot\tau) = \{0\}$.

Proof: For $\varphi \in \mathrm{Hom}_G(\sigma, m\cdot\tau)$, letting $f = \frac{\dim\sigma}{|G|}\cdot\chi_{\tau^\vee}$,

$$\varphi(v) \;=\; f\cdot\varphi(v) \;=\; \varphi(f\cdot v) \;=\; \varphi(0) \;=\; 0 \qquad\qquad (\text{for } v\in\sigma)$$

since $\mathrm{Hom}_G(\tau,\sigma) = \{0\}$. ///

Corollary: In a decomposition of finite-dimensional unitary G-representation V as a finite orthogonal direct sum of G-irreducibles, the number of occurrences of each irreducible π is *uniquely determined*, namely $\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi, V)$. ///

Remark: The decomposition into *isotypes* is canonical, given by application of (normalized) χ_π for π irreducible. The decomposition into *irreducibles* is not quite canonical, because $\dim \mathrm{Hom}_G(m\sigma, m\sigma) = m^2$.

Now Hecke's identity can be proven.

The space $V = \mathbb{C}[x_1, \ldots, x_n]^{\leq d}$ of all polynomials of total degree at most d is stable under $SO(n, \mathbb{R})$, and $P \to P^\#$ is an $SO(n, \mathbb{R})$-map of it to itself.

We have shown that

$$V \;=\; H_d \oplus H_{d-1} \oplus r^2 H_{d-2} \oplus H_{d-2} \oplus r^2 H_{d-3} \oplus H_{d-3}$$
$$\oplus\, r^4 H_{d-4} \oplus r^2 H_{d-4} \oplus H_{d-4} \oplus \ldots$$

Since $SO(n, \mathbb{R})$ acts trivially on r, as $SO(n, \mathbb{R})$-spaces $r^m H_{d-j} \approx H_{d-j}$. Thus,

$$V \approx (1 \cdot H_d) \oplus (1 \cdot H_{d-1}) \oplus (2 \cdot H_{d-2}) \oplus (2 \cdot H_{d-3}) \oplus (3 \cdot H_{d-4}) \oplus \ldots$$

Δ^S commutes with $SO(n, \mathbb{R})$, and is $\lambda_d = -d(d+n-2)$ on H_d. Thus, $H_d \not\approx H_{d-j}$ for $j > 0$ ... *if* Δ^S has an intrinsic sense!

And, yes, it does, but *not* via the $S^{n-1} \subset \mathbb{R}^n$ description!

Granting this for a moment, there is a single copy of H_d in V, which # maps to itself, by a *scalar*.

The scalar can be determined by evaluating # on a single $P(x)e^{-\pi|x|^2}$. Conveniently, we can systematically write explicit harmonic polynomials of all degrees: $(x+iy)^d e^{-\pi|x|^2} = z^d e^{-\pi z\overline{z}}$. Direct computation shows that the eigenvalue is i^{-d}. The proves Hecke's identity. ///

It remains to show that Δ^S is *intrinsic*...