• Intrinsic-ness of $SO(n, \mathbb{R})$-invariant Laplacian Δ^S

Calculus on spheres is the simplest non-Euclidean example.

Euclidean calculus on products of circles and lines, and the corresponding harmonic analysis, is the *archimedean* part of much basic number theory.

Calculus on non-Euclidean spaces is even more useful, but requires more preparation.

The *spherical*-geometry case is significantly easier than *hyperbolic*-geometry examples, and simpler than more general situations.

Hecke's identity on Fourier transforms of harmonic-polynomial multiples of Gaussians is a good excuse to introduce a bit of representation theory, especially of compact groups.

What remains is giving an *intrinsic* meaning to Δ^S.

Let G be a subgroup of the group $GL_n(\mathbb{R})$ of multiplicatively-invertible n-by-n real matrices.

We are only concerned with *very nice* subgroups G, probably requiring that G is defined by *polynomial conditions* on the entries of the matrices.

For example, the rotation group (special orthogonal group) $SO(n, \mathbb{R})$ is defined by the collection of quadratic equations arising from the defining condition $g^\top g = 1_n$.

Another example is $SL_n(\mathbb{R}) = \{g \in GL_n(\mathbb{R}) : \det g = 1\}$.

These conditions are topologically *closed*, so such groups are closed subgroups of $GL_n(\mathbb{R})$.

The rotation group $SO(n, \mathbb{R})$ is *compact*.

For non-abelian groups, the distinction between *position* and *direction* that can be overlooked in $\mathbb{R}^n$ becomes enormous. Specifically, specifying *direction* (and *directional derivatives*) requires more, as follows.

The *matrix exponential* is given by the expected series

$$e^A = \exp(A) = \sum_{i \geq 0} \frac{A^i}{i!} \qquad \text{(for } n\text{-by-}n \text{ matrix } A\text{)}$$

The n-by-n real or complex matrices form a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, so have a unique (reasonable) *topology*, although the topology can be described by several different *norms*. The exponential series converges absolutely for any such norm.

When $AB = BA$, we do have $e^{A+B} = e^A \cdot e^B$ by the usual argument, but when A and B do not commute this identity *fails*.

For a nice $G \subset GL_n(\mathbb{R})$, the *Lie algebra* of G will give *directional derivatives* (=tangent vectors) on G. One definition is

$$\text{Lie}\, G \;=\; \mathfrak{g} \;=\; \{n\text{-by-}n\ A \;:\; e^{tA} \in G \text{ for all } t \in \mathbb{R}\}$$

Claim:

$$\mathfrak{gl}_n(\mathbb{R}) \quad = \quad \text{Lie}\, GL_n(\mathbb{R}) \quad = \quad \{\text{all } n\text{-by-}n\ A\}$$

$$\mathfrak{sl}_n(\mathbb{R}) \quad = \quad \text{Lie}\, SL_n(\mathbb{R}) \quad = \quad \{n\text{-by-}n\ A \text{ with } \mathrm{tr} A = 0\}$$

$$\mathfrak{so}(n,\mathbb{R}) \quad = \quad \text{Lie}\, SO(n,\mathbb{R}) \quad = \quad \{A + A^\top = 0 \text{ and } \mathrm{tr} A = 0\}$$

Proof: Using Jordan form, $\det e^A \;=\; e^{\mathrm{tr} A}$. Thus, $\det e^A \;\neq\; 0$, so is invertible for all A. For $\mathrm{tr} A \;=\; 0$, $\det e^A \;=\; e^0 \;=\; 1$. For the orthogonal group, ...

... first observe that $(e^A)^\top = e^{A^\top}$. Thus, the condition $(e^{tA})^\top e^{tA} = 1_n$ is

$$(1_n + tA^\top + \ldots) \cdot (1_n + tA + \ldots) \;=\; 1_n$$

The linear-in-t term is $A^\top + A = 0_n$, so this condition is *necessary*. With $A^\top = -A$,

$$(e^{tA})^\top e^{tA} \;=\; e^{tA^\top} \cdot e^{tA} \;=\; e^{-tA} \cdot e^{tA} \;=\; e^0 \;=\; 1_n$$

proving *sufficiency*. ///

The derivative $X_A f$ of a smooth function f on G in direction $A \in \mathfrak{g}$ is

$$(X_A f)(g) \;=\; \frac{\partial}{\partial t}\Big|_{t=0} f(ge^{tA})$$

These differential operators do not commute: the commutator turns out to be

$$[X_A, X_B] \;=\; X_A \circ X_B - X_B \circ X_A \;=\; X_{AB-BA} \;=\; X_{[A,B]}$$

for $A, B \in \mathfrak{g}$. Always $[A, B] \in \mathfrak{g}$ for $A, B \in \mathfrak{g}$. This is *suggested* by

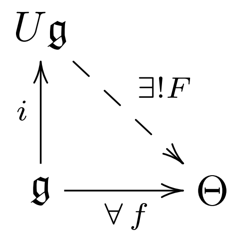
$$e^{tA}e^{tB}e^{-tA}e^{-tB} \;=\; (1 + tA + \frac{t^2A^2}{2} + \ldots)(1 + tB + \frac{t^2B^2}{2} + \ldots)$$

$$\times(1 - tA + \frac{t^2A^2}{2} + \ldots)(1 - tB + \frac{t^2B^2}{2} + \ldots)$$

$$= \; 1 + t^2(AB - BA) + \ldots$$

While commutators $[X_A, X_B]$ *do* arise from $[A, B] \in \mathfrak{g}$, simple *compositions* $X_A \circ X_B$ do not come from anything in $\mathfrak{g}$. This awkwardness is remedied as follows.

We want an associative $\mathbb{C}$-algebra $U\mathfrak{g}$ and a $[,]$-preserving map $i : \mathfrak{g} \to U\mathfrak{g}$ such that, for all $[,]$-preserving maps $f : \mathfrak{g} \to \Theta$ to an associative $\mathbb{C}$-algebra Θ, there is a unique associative algebra map $U\mathfrak{g} \to \Theta$ through which f factors. That is, we have



This characterizes $i : \mathfrak{g} \to U\mathfrak{g}$ uniquely up to unique isomorphism, *if it exists.* In other words, the functor U that creates $U\mathfrak{g}$ from $\mathfrak{g}$ is *adjoint* to the (forgetful) functor *Lie* that creates $[x, y] = xy - yx$ on an associative algebra A:

$$\mathrm{Hom}_{\mathrm{assoc}}(U\mathfrak{g}, A) \;\approx\; \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Lie}\, A)$$

For existence, in fact, $U\mathfrak{g}$ is a quotient of the *universal associative algebra* $A\mathfrak{g}$ attached to the *vector space* $\mathfrak{g}$, forgetting the $[,]$ structure, and requiring that all vector space maps $\mathfrak{g} \to V$ factor through an associative algebra map $A\mathfrak{g} \to V$.

The universal associative algebra is often constructed as

$$A\mathfrak{g} \;=\; \bigotimes^\bullet \mathfrak{g} \;=\; \mathbb{C} \oplus \bigoplus_{n \geq 1} \Big(\bigotimes^n \mathfrak{g} \Big)$$

which has the boring/universal multiplication

$$(x_1 \otimes \ldots \otimes x_m) \cdot (y_1 \otimes \ldots \otimes y_n) \;=\; x_1 \otimes \ldots \otimes x_m \otimes y_1 \otimes \ldots \otimes y_n$$

The lack of interesting or special features here is exactly the universality. The enveloping algebra $U\mathfrak{g}$ is the quotient by the ideal generated by all $x \otimes y - y \otimes x - [x, y]$. The map $\mathfrak{g} \to U\mathfrak{g}$ is induced from $\mathfrak{g} \to \bigotimes^1 \mathfrak{g} \subset \bigotimes^\bullet \mathfrak{g}$.

The image of the tensor $x_1 \otimes \ldots \otimes x_m$ in $U\mathfrak{g}$ is simply written without the tensor symbols, namely, $x_1 \ldots x_m$.

Whenever the action of G on a representation space π is *differentiable*, the Lie algebra $\mathfrak{g}$ acts by

$$Av \;=\; \frac{\partial}{\partial t}\Big|_{t=0} \pi(e^{tA})(v) \qquad\qquad (\text{for } A \in \mathfrak{g} \text{ and } v \in \pi)$$

This map $\mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\pi)$ preserves brackets (!), so gives a unique corresponding associative-algebra map $U\mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\pi)$.

All these actions are compatible with the action of G, since they are induced by it. For example,

$$\Big(\pi(g) \circ A \circ \pi(g)^{-1}\Big)(v) \;=\; (gAg^{-1})(v)$$

Here, gAg^{-1} is simply matrix conjugation, but/and it has an abstract sense in general, and is called the *Adjoint action* of G on $\mathfrak{g}$, denoted $\mathrm{Ad}\, g(A)$. The lower-case *adjoint action* of $\mathfrak{g}$ on itself is by $\mathrm{ad}\, x(y) = [x, y]$. The Adjoint action of G on $\mathfrak{g}$ gives rise to a natural action on $\bigotimes^\bullet \mathfrak{g}$ and $U\mathfrak{g}$.

There is a possibly-unexpected advantage to considering the universal enveloping algebra, namely, that the G-fixed subalgebra $\mathfrak{z} = (U\mathfrak{g})^G$ is quite non-trivial!!!

The simplest non-scalar element in $\mathfrak{z}$ is the *Casimir* element Ω, described as follows. The *trace form* is $\langle A, B \rangle = \mathrm{tr}(AB)$. This is a non-degenerate, symmetric, AdG-invariant pairing on the Lie algebras $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{so}(n, \mathbb{R})$, and many others.

Up to a normalizing constant, $\langle , \rangle$ is the *Killing form*, not because it kills anything, but because of pioneering work by Wilhelm Killing.

For $\mathfrak{so}(n, \mathbb{R})$, the trace form is *positive definite*, as is clear from noting the orthogonal basis $\theta_{ij} = e_{ij} - e_{ji}$ for $i < j$, where e_{ij} has non-zero entry only at the ij^{th} location, with a 1 there.

The non-degenerate pairing on $\mathfrak{g}$ gives a natural identification of $\mathfrak{g}$ with its dual $\mathfrak{g}^*$, by $\lambda_A(B) = \langle A, B \rangle$.

Consider the natural, G-equivariant maps

$$\mathrm{End}_{\mathbb{C}}(\mathfrak{g}) \;\approx\; \mathfrak{g} \otimes \mathfrak{g}^* \;\approx\; \mathfrak{g} \otimes \mathfrak{g} \;\subset\; \bigotimes{}^\bullet \mathfrak{g} \;\longrightarrow\; U\mathfrak{g}$$

At the left end, the identity endomorphism $1_{\mathfrak{g}}$ certainly commutes with $\mathrm{Ad}\, G$. With a choice of basis x_i for $\mathfrak{g}$ and dual basis x_i^*, the image of $1_{\mathfrak{g}}$ in $g \otimes \mathfrak{g}^*$ is $\sum_i x_i \otimes x_i^*$. Of course, noting that this is the image of $1_{\mathfrak{g}}$, it is *a-fortiori* G-invariant.

Let Ω be the image of $1_{\mathfrak{g}}$ in $U\mathfrak{g}$. By design, $\Omega \in (U\mathfrak{g})^G$, but it is not entirely clear that it is not accidentally 0.

For any choice of basis x_i and dual basis x_i^*, $\Omega = \sum_i x_i x_i^*$.

Remark: Some sources perversely *define* Ω by the formula in terms of a basis and dual basis, and then prove that the expression is invariant under change-of-basis. It is obviously better to give an intrinsic definition that does not refer to a basis.

Non-Euclidean geometries attached to G often have *Laplacian* given by the corresponding Casimir element.

By design, the action of Ω on a representation space is *intrinsic*, depending only on the isomorphism class. Since Ω commutes with G, by Schur's Lemma it acts by a *scalar* on an irreducible.

For $G = SO(n, \mathbb{R})$, let's compute the effect of Ω on H_d, using the orthogonal basis $\theta_{ij} = e_{ij} - e_{ji}$. All these are length $\sqrt{2}$, so $\Omega = \frac{1}{2} \sum_{i<j} (e_{ij} - e_{ji})^2 \in \mathfrak{z} \subset U\mathfrak{g}$.

Of course, $\exp \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$, so

$$\exp t\theta_{ij} = \begin{pmatrix} 1 & & & & \\ & \cos t & & \sin t & \\ & & 1 & & \\ & -\sin t & & \cos t & \\ & & & & 1 \end{pmatrix}$$

With $G = SO(n,\mathbb{R})$ acting on functions f on S^{n-1} by $g \cdot f(x) = f(xg)$, the summand $\theta_{ij}\theta_{ij}$ acts by

$$\frac{\partial^2}{\partial t^2}\Big|_{t=0} f(x \cdot e^{t\theta_{ij}})$$

$$= \frac{\partial^2}{\partial t^2}\Big|_{t=0} f(\ldots, x_i \cos t - x_j \sin t, \ldots, x_i \sin t + x_j \cos t, \ldots)$$

with something non-trivial only at i^{th} and j^{th} arguments. This is

$$\frac{\partial}{\partial t}\Big|_{t=0}\Big((-x_i \sin t - x_j \cos t) f_i + (x_i \cos t - x_j \sin t) f_j\Big)$$

$$= -x_i f_i + x_j^2 f_{ii} + x_i^2 f_{jj} - x_j f_j - 2x_i x_j f_{ij}$$

For *homogeneous* f of total degree d, Euler's $\sum_i x_i f_i = d \cdot f$, and $\sum_{ij} x_i x_j f_{ij} = d(d-1)f$ help simplify:

for f homogeneous of total degree d,

$$(\Omega f)(x) \;=\; \sum_{i<j} \Big(-x_i f_i + x_j^2 f_{ii} + x_i^2 f_{jj} - x_j f_j - 2x_i x_j f_{ij} \Big)$$

$$=\; -(n-1)d\cdot f + \sum_{i<j} \Big(x_j^2 f_{ii} + x_i^2 f_{jj} - 2x_i x_j f_{ij} \Big)$$

$$=\; -(n-1)d\cdot f + \tfrac{1}{2} \sum_{i\neq j} \Big(x_j^2 f_{ii} + x_i^2 f_{jj} - 2x_i x_j f_{ij} \Big)$$

$$=\; -(n-1)d\cdot f \;+\; r^2 \Delta f \;-\; d(d-1)\cdot f \;=\; -d(d+n-2)\cdot f \;+\; r^2\Delta f$$

Seemingly-miraculously, for *harmonic* f of total degree d, this recovers the eigenvalue for the *extrinsic* Δ^S, essentially giving $\Omega f = \Delta^S f$.

That is, those eigenvalues are not mere artifacts! They are intrinsic, so depend only on the isomorphism class of the representation, and our argument for Hecke's identity is complete. ///