(More supporting details clarifying Iwasawa-Tate theory would be useful, but there are higher priorities...)

• Classfield Theory

In brief, *global* classfield theory classifies *abelian* extensions of *number fields* and function fields. Takagi and Artin, independently, about 1928.

In brief, *local* classfield theory does the analogous things for *local fields*: finite extensions of $\mathbb{Q}_p$ and of $\mathbb{F}_p((x))$.

The details of these classifications also subsume all known (abelian) **reciprocity laws**.

Non-abelian classfield theory is an almost entirely *conjectural* extension of (genuine) classfield theory to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Proofs are much harder than *statements*, and there are various levels of sophistication of the statements themselves.

Main Theorem of Global Classfield Theory (classical form): The abelian (Galois) extensions K of a number field k are in bijection with generalized ideal class groups, which are quotients of *ray class groups* of *conductor* (a non-zero ideal) $\mathfrak{f}$

$$I(\mathfrak{f})/P_{\mathfrak{f}}^{+}$$

$$||$$

$$\frac{\text{fractional ideals prime to } \mathfrak{f}}{\text{principal ideals with totally positive generators 1 mod } \mathfrak{f}}$$

Further, the bijection sends a given generalized ideal class group to the (abelian) *Galois group* of the extension, via the *Artin/Frobenius* map/symbols $\mathfrak{p} \to (\mathfrak{p}, K/k)$ [see below].

Main Theorem of Local Classfield Theory: The abelian (Galois) extensions K of a local field k are in bijection with the open, finite-index subgroups of $k^\times$, by

$$K/k \longleftrightarrow k^\times/N_k^K K^\times$$

This bijection is given by an isomorphism of the Galois group with $k^\times/N_k^K K^\times$ via Artin/Frobenius.

Remark: Historically, local classfield theory was proven as a corollary of global classfield theory, using the idea that, given an abelian extension of a local field, there is an extension K/k of *number* fields and completions k_v of k and K_w of K such that K_w/k_v is the given extension of local fields.

Global classfield theory subsumes reciprocity laws:
(at least ... *abelian* ones). This includes

Quadratic reciprocity over $\mathbb{Q}$ (Gauss, 1800) and over arbitrary number fields and function fields (Takagi, Artin, 1928, Iwasawa-Tate 1950, Weil 1964)

Equivalently, the zeta function of a quadratic extension of $\mathbb{Q}$ is the product of zeta of $\mathbb{Q}$ with a quadratic-character L-function.

Cubic and biquadratic reciprocity (Gauss, Jacobi, Eisenstein: 1820-44)

Factorization of zeta-functions of cyclotomic fields as products of Dirichlet L-functions over $\mathbb{Q}$ (Dirichlet, 1837).

Factorization of zeta-functions of abelian extensions of $\mathbb{Q}$ as products of Dirichlet L-functions over $\mathbb{Q}$.

One further aspect of classfield theory: One original technical motivation of further extensions of a given number field was to try to make non-principal ideals become principal.

The **Hauptidealsatz** (*Principal ideal theorem*) is the assertion that all ideals in $\mathfrak{o}_k$ become principal in the abelian extension of k corresponding to the *absolute* ideal class group. (Conjectured by Hilbert about 1897, proven by Furtwangler 1930). Examples had been known to Kummer and Kroncker.

The abelian extension corresponding to the absolute ideal class group is the *Hilbert classfield*, and is the *maximal unramified* abelian extension of the base: as part of global classfield theory, the only possible ramification is at primes dividing the conductor, for the absolute ideal class group just 1.

Golod and Shafarevich 1964 showed that there exist *infinite classfield towers*, meaning that it is futile to go to larger-and-larger fields hoping to find a PID.

Another technical aspect: norms in cyclic extensions

A key point in (every version of) *proof* of global classfield theory is

Cyclic local-global principle for norms: In a *cyclic* extension K/k of number fields, an element of k is a *global norm* if and only if it is a *local norm everywhere.* For $\alpha \in k$,

$$\alpha \in N_k^K(K^\times) \iff \alpha \in N_{k_v}^{K_w}(K_w^\times) \text{ for all } v, w$$

The most intelligible proof of *this* is probably Weil's 1967 rewrite of Noether's pre-1940 ideas, encapsulating some cohomological ideas in structure of semi-simple algebras. The Artin-Tate notes from 1952 are more overtly cohomological.