- **Classfield Theory...**
- Herbrand quotient as Euler-Poincaré characteristic
- Toward Hilbert's theorem 90 as cohomology *cont'd*
- Toward classfield theory of cyclic extensions of local fields

Again, the early conceptions of classfield theory, from reciprocity laws of Gauss 1796, Eisenstein, Jacobi, through Kummer and Kronecker, the relative-quadratic examples of Hilbert 1897, Takagi's and Artin's proofs in the 1920s and 1930s, were substantial enough that there was little concern for *rewriting.*

Nevertheless, with hindsight gained from a decade of application of Noether's abstract algebra to algebraic topology, by the late 1930s Chevalley, Weil, and others could see the possibility of usefully rewriting classfield theory overtly using the cohomological ideas that had been lurking inside it.

Herbrand quotients: less-bare definition An abelian group A with an ordered pair of maps $f : A \to A$ and $g : A \to A$, with $f \circ g = 0$ and $g \circ f = 0$ gives a periodic *complex*

$$\cdots \xrightarrow{f} A \xrightarrow{g} A \xrightarrow{f} A \xrightarrow{g} \cdots$$

This is an example of a *complex*

$$\cdots \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} \cdots$$

where the essential requirement is that the composition $f_{i+1} \circ f_i$ of any two successive maps is 0, that is, that $\ker f_i \subset \operatorname{im} f_{i+1}$.

The *(co-) homology* of the complex is the collection of quotients

$$H_i(\text{the complex}) \;=\; H^i(\text{the complex}) \;=\; \frac{\ker f_i|_{A_i}}{\operatorname{im} f_{i-1}|_{A_{i-1}}}$$

The periodic complex

$$\cdots \xrightarrow{f} A \xrightarrow{g} A \xrightarrow{f} A \xrightarrow{g} \cdots$$

has just two (co-) homology groups,

$$\frac{\ker f|_A}{\operatorname{im} g_A} \qquad\qquad \frac{\ker g|_A}{\operatorname{im} f_A}$$

and there is no natural indexing. The Herbrand quotient is the ratio of the orders of these groups:

$$\text{Herbrand quotient of } A, f, g \;=\; q_{f,g}(A) \;=\; \frac{[\ker f : \operatorname{im} g]}{[\ker g : \operatorname{im} f]}$$

Inscrutable Key Lemma: For finite A, $q(A) = 1$. For f-stable, g-stable subgroup $A \subset B$ with $f, g : B \to B$, we have $q(B) = q(A) \cdot q(B/A)$, in the usual sense that if two are finite, so is the third, and the relation holds. *(Proof below)*

In fact, letting $C = B/A$, the lemma refers to a situation

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}$$

with columns *complexes* and rows *exact*, where again,

$\cdots \xrightarrow{f} X \xrightarrow{g} \cdots$ *exact* means $\ker g = \operatorname{im} f$.

Important special cases are that $0 \to A \to B$ implies $A \to B$ *injects*, and $B \to C \to 0$ implies $B \to C$ *surjects*.

The latter diagram is *commutative*, in the sense that compositions of maps are independent of the route through the diagram.

More precisely, recall that a diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[i]{} & D \end{array}$$

is *commutative* when the composition along the upper right is equal to the composition along the lower left, that is, $h \circ f = i \circ g$.

In the Herbrand quotient diagram, a special case of the **long exact sequence in (co-) homology** will give a periodic long exact sequence

$$\ldots \to \frac{\ker f_A}{\operatorname{im} g_A} \to \frac{\ker f_B}{\operatorname{im} g_B} \to \frac{\ker f_C}{\operatorname{im} g_C} \to \frac{\ker g_A}{\operatorname{im} f_A} \to \frac{\ker g_B}{\operatorname{im} f_B} \to \frac{\ker g_C}{\operatorname{im} f_C} \to \ldots$$

The periodicity often is emphasized by writing the long exact sequence as

$$
\begin{array}{ccccc}
 & \dfrac{\ker f|_A}{\operatorname{im} g|_A} & \longrightarrow & \dfrac{\ker f|_B}{\operatorname{im} g|_B} & \\
\nearrow & & & & \searrow \\
\dfrac{\ker g|_C}{\operatorname{im} f|_C} & & & & \dfrac{\ker f|_C}{\operatorname{im} g|_C} \\
\nwarrow & & & & \swarrow \\
 & \dfrac{\ker g|_B}{\operatorname{im} f|_B} & \longleftarrow & \dfrac{\ker g|_A}{\operatorname{im} f|_A} &
\end{array}
$$

The numerical assertion of the Herbrand lemma is extracted from this periodic exact sequence by *Euler-Poincaré characteristics.*

Claim: (Protype) The *Euler characteristic* $\sum_i (-1)^i \dim F_i$ of an exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \ldots \longrightarrow V_{n-1} \longrightarrow V_n \longrightarrow 0$$

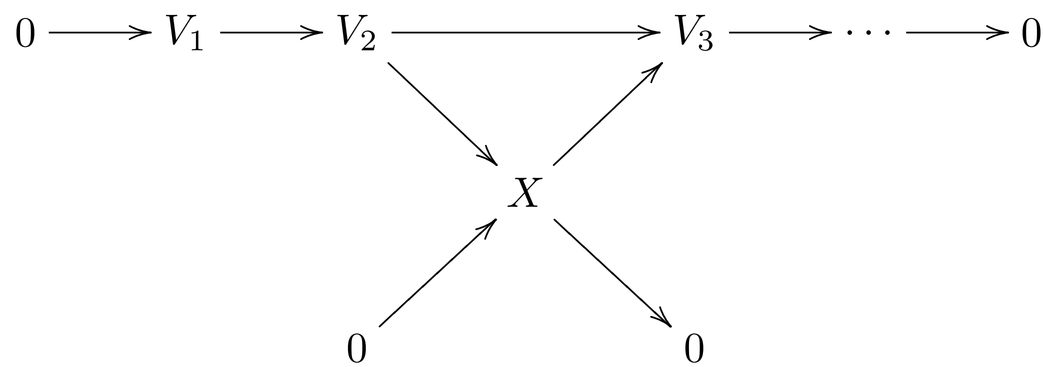
of vector spaces over a field is 0.

Proof: (Recap) For a *short* exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ of vector spaces, the standard idea that any basis of V_1 can be extended to a basis of V_2, with the (images of the) *new* elements forming a basis of $V_3 \approx V_2/V_1$, proves the assertion in this case.

The general case is by induction: an exact sequence

$$0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_{n-1} \longrightarrow V_{n-1} \longrightarrow V_n \longrightarrow 0$$

with $n > 3$ can be *spliced* together from two smaller ones:



with $X = \mathrm{im}\, V_2 = \ker(V_3 \to V_4)$, using exactness.

That is, we have exact

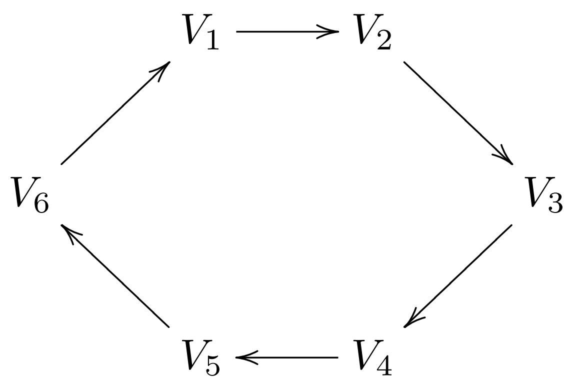
$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow X \longrightarrow 0$$

and

$$0 \longrightarrow X \longrightarrow V_3 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

Add the corresponding equations $\dim V_1 - \dim V_2 + \dim X = 0$ and (by induction) $\dim X - \dim V_3 + ... + (-1)^n \dim V_n = 0$. ///

Corollary: The Euler-Poincaré characteristic $\dim V_1 - \dim V_2 + \dim V_3 - \dim V_4 + \dim V_5 - \dim V_6$ of a *periodic* exact diagram of vector spaces



is 0.

Proof: Use the splicing trick, with

$$X = \ker(V_1 \to V_2) = \mathrm{im}\,(V_6 \to V_1)$$

to rewrite the periodic exact sequence as

$$0 \longrightarrow X \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_6 \longrightarrow X \longrightarrow 0$$

The Euler-Poincaré characteristic of the un-spliced exact sequence is

$$0 = (-1)^1 \dim X - \Big(\sum_{i=1}^{6} (-1)^i \dim V_i\Big) + (-1)^8 \dim X$$

$$= -\sum_{i=1}^{6} (-1)^i \dim V_i$$

giving the asserted vanishing. ///

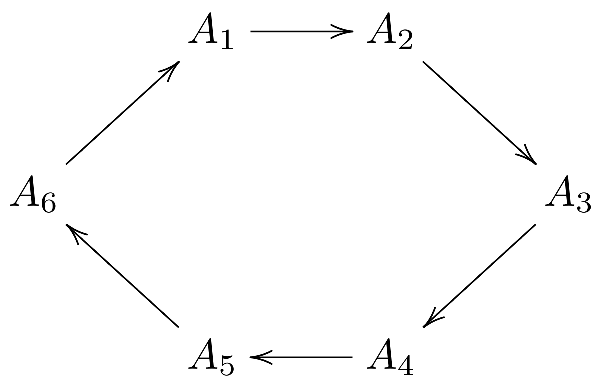
Remark: By the same arguments, for exact sequences of *finite* abelian groups

$$0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_{n-1} \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow 0$$

we have

$$\frac{|A_1| \cdot |A_3| \cdot |A_5| \cdot \ldots}{|A_2| \cdot |A_4| \cdot |A_6| \ldots} \;=\; 1$$

and the analogous corollary: for periodic exact



we have

$$\frac{|A_1| \cdot |A_3| \cdot |A_5|}{|A_2| \cdot |A_4| \cdot |A_6|} \;=\; 1$$

In the periodic exact sequence

$$\begin{array}{ccccccc} & & \dfrac{\ker f|_A}{\operatorname{im} g|_A} & \longrightarrow & \dfrac{\ker f|_B}{\operatorname{im} g|_B} & & \\ & \nearrow & & & & \searrow & \\ \dfrac{\ker g|_C}{\operatorname{im} f|_C} & & & & & & \dfrac{\ker f|_C}{\operatorname{im} g|_C} \\ & \nwarrow & & & & \swarrow & \\ & & \dfrac{\ker g|_B}{\operatorname{im} f|_B} & \longleftarrow & \dfrac{\ker g|_A}{\operatorname{im} f|_A} & & \end{array}$$

group the cardinalities belonging to A, B, C, and note the inversion for B:

$$1 \quad = \quad \frac{|A_1|}{|A_4|} \cdot \frac{|A_5|}{|A_2|} \cdot \frac{|A_3|}{|A_6|}$$

$$= \frac{[\ker f_A : \operatorname{im} g_A]}{[\ker g_A : \operatorname{im} f_A]} \cdot \frac{[\ker g_B : \operatorname{im} f_B]}{[\ker f_B : \operatorname{im} g_B]} \cdot \frac{[\ker f_C : \operatorname{im} g_C]}{[\ker g_C : \operatorname{im} f_C]} \qquad ///$$

Remark: The finiteness assertions were omitted, but it is clear that the Herbrand quotient lemma is a corollary of Euler-Poincaré characteristic ideas and the *long exact sequence in homology*.

Theorem: *(shortest long exact sequence)* A commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow f & & \downarrow f & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact *rows* gives a long exact sequence

$$0 \to \ker f|_A \to \ker f|_B \to \ker f|_C \to \frac{A'}{fA} \to \frac{B'}{fB} \to \frac{C'}{fC} \to 0$$

Remark: The diagram is a short exact sequence of the *complexes* $0 \to A \to A' \to 0$, $0 \to B \to B' \to 0$, and $0 \to C \to C' \to 0$.

Least obvious part of the proof: The *connecting homomorphism* $\delta : \ker f|_C \longrightarrow A'/fA$ is not obvious. Recopying the diagram,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Given $f(c) = 0$, take $b \to c$, by surjectivity of $B \to C$. Then $f(b) \to f(c) = 0$, so $f(b)$ is in the kernel of $B' \to C'$. By exactness of $A' \to B' \to C'$, there is $a' \to f(b)$. Put $\delta(c) = a'$. (The rest of the proof is more natural.) ///

Remark: The *Snake Lemma* is the description of the connecting homomorphism. *There is non-trivial content in its existence.*

Example: Euler's integral $\Gamma(s) = \int_0^\infty t^s\, e^{-t}\, \frac{dt}{t}$ converges for $\mathrm{Re}(s) > 0$. The usual way to see that this has an *meromorphic continuation* is to repeatedly integrate by parts.

However, the long exact sequence in homology shows that the values are completely determined, in any case!

Rewrite the integral as an integral over the whole line, by replacing t by x^2:

$$\Gamma(s) = \int_0^\infty t^s\, e^{-t}\, \frac{dt}{t} = \int_{\mathbb{R}} |x|^{2s-1}\, e^{-x^2}\, dx$$

The Gaussian e^{-x^2} is in the Schwartz space $\mathscr{S}$ on $\mathbb{R}$, and for $\mathrm{Re}(\lambda) > 0$ the map $u_\lambda(\varphi) = \int_{\mathbb{R}} |x|^\lambda\, \varphi(x)\, dx$ is in the space $\mathscr{S}^*$ of continuous linear functionals on $\mathscr{S}$, that is, *tempered distributions.*

u_λ can be meromorphically continued *as a tempered-distribution-valued function of* λ. Strikingly, without meromorphic continuation, u_λ is determined by the Snake Lemma, that is, by the long exact sequence in homology, as follows.

Observe that for $\mathrm{Re}(\lambda) \gg 1$, u_λ is differentiable, and $xu'_\lambda = \lambda \cdot u_\lambda$. That is, for such λ, u_λ is annihilated by

$$T_\lambda \;=\; x\frac{d}{dx} - \lambda$$

Let $\mathscr{S}_o$ be the space of Schwartz functions *vanishing to infinite order* at 0, and $\mathscr{S}_o^*$ its dual.

Let v_λ be u_λ restricted to $\mathscr{S}_o$, where the integral converges for *all* $\lambda \in \mathbb{C}$. That is, v_λ is *entire* as a function of λ.

We wish to *extend* v_λ from $\mathscr{S}_o$ to S, thus *continuing* u_λ outside the region of convergence of the integral.

Characterize u_λ and v_λ as being solutions of the equation $T_\lambda u = 0$.

Thus, in the surjection $\mathscr{S}^* \to \mathscr{S}_o^*$, we want $u_\lambda \in \mathscr{S}^*$ mapping to v_λ *and* $u_\lambda \in \ker T_\lambda$. Further, u_λ should be *unique*.

$X = \ker(\mathscr{S}^* \to \mathscr{S}_o^*)$ consists of distributions supported at 0. By the theory of Taylor-Maclaurin expansions, X is finite linear combinations of Dirac δ and its derivatives. Consider the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & \mathscr{S}^* & \longrightarrow & \mathscr{S}_o^* & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle T_\lambda} & & \downarrow {\scriptstyle T_\lambda} & & \downarrow {\scriptstyle T_\lambda} & & \\ 0 & \longrightarrow & X & \longrightarrow & \mathscr{S}^* & \longrightarrow & \mathscr{S}_o^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

We have $v_\lambda \in \ker T_\lambda\big|_{\mathscr{S}_o^*}$, and want to find *unique* $u_\lambda \in \ker T_\lambda\big|_{\mathscr{S}^*}$ surjecting to v_λ. This is exactly what the long exact sequence gives a criterion for:

The not-so-long long exact sequence is

$$0 \longrightarrow \ker T_\lambda|_X \longrightarrow \ker T_\lambda\Big|_{\mathscr{S}_o^*} \longrightarrow \ker T_\lambda\Big|_{\mathscr{S}^*}$$

$$\longrightarrow \frac{X}{T_\lambda X} \longrightarrow \frac{\mathscr{S}_o^*}{T_\lambda \mathscr{S}_o^*} \longrightarrow \frac{\mathscr{S}^*}{T_\lambda \mathscr{S}^*} \longrightarrow 0$$

The part of interest is

$$0 \longrightarrow \ker T_\lambda|_X \longrightarrow \ker T_\lambda\Big|_{\mathscr{S}_o^*} \longrightarrow \ker T_\lambda\Big|_{\mathscr{S}^*} \longrightarrow \frac{X}{T_\lambda X}$$

Thus, $v_\lambda \in \ker T|_{\mathscr{S}_o^*}$ is assured to be in the image of $\ker T_\lambda|_{\mathscr{S}^*}$ when $X/T_\lambda X = 0$, and *uniquely* so exactly when $\ker T_\lambda|_X = 0$.

Remark: We reach these conclusions without knowing the details of the *connecting homomorphism*, or any of the other (more elementary) maps.

Thus, the desired u_λ *certainly exists* when $X/T_\lambda X = 0$, that is, when $T_\lambda X = X$, and *uniquely* so *exactly* when $T_\lambda u = 0$ has no non-trivial solution in X.

We compute that for test function φ

$$(x\frac{d}{dx}\delta)(\varphi) \;=\; (\frac{d}{dx}\delta)(x\varphi) \;=\; -\delta(\frac{d}{dx}x\varphi)$$

$$= \; -x\Big|_{x=0} \cdot \varphi'(0) - \frac{dx}{dx}\Big|_{x=0} \cdot \varphi(0) \;=\; \varphi(0) \;=\; \delta(\varphi)$$

That is, $x\frac{d}{dx}\delta = -\delta$. By induction, $x\frac{d}{dx}\delta^{(n)} = -(n+1)\cdot\delta^{(n)}$.

Thus, u_λ exists and is unique for $\lambda \not\in \{-1,-2,-3,\ldots\}$. Thus, $\Gamma(s) = u_{2s-1}(e^{-x^2})$ *certainly* exists for $s \not\in \{0,-\frac{1}{2},-1,-\frac{3}{2},-2,\ldots\}$.

Remark: This incorrectly indicates potential trouble at negative half-integers. There is no such trouble, further information about the maps in the long exact sequence is needed.