- **Classfield Theory...**
- Herbrand quotient as Euler-Poincaré characteristic
- More-elementary kernel-image relations
- Norm index equality for cyclic extensions of local fields

Although we can produce long exact sequences in (co-) homology from short exact sequences of *complexes*, we have no general meaningful mechanism to produce those short exact sequences.

An obvious example is the short exact sequence of *complexes* produced from a short exact sequence $0 \to A \to B \to C \to 0$ of *modules*, by the universal complex construction

$$A \quad \longrightarrow \quad \Big(\cdots \xrightarrow{t} A \xrightarrow{\sigma-1} A \xrightarrow{t} A \xrightarrow{\sigma-1} \cdots \Big)$$

for any G-module A, where $G = \langle\sigma\rangle$ is finite cyclic, $t = \sum_{g\in G} g$, and we know $(\sigma - 1) \circ t = t \circ (\sigma - 1) = 0$.

Herbrand quotients: less-bare definition An abelian group A with an ordered pair of maps $f : A \to A$ and $g : A \to A$, with $f \circ g = 0$ and $g \circ f = 0$ gives a periodic *complex*

$$\cdots \xrightarrow{f} A \xrightarrow{g} A \xrightarrow{f} A \xrightarrow{g} \cdots$$

with just two (co-) homology groups,

$$\frac{\ker f|_A}{\operatorname{im} g_A} \qquad \frac{\ker g|_A}{\operatorname{im} f_A}$$

and no natural indexing. The Herbrand quotient is the ratio of the orders of these groups:

$$\text{Herbrand quotient of } A, f, g \;=\; q_{f,g}(A) \;=\; \frac{[\ker f : \operatorname{im} g]}{[\ker g : \operatorname{im} f]}$$

Key Lemma: For finite A, $q(A) = 1$. For f-stable, g-stable subgroup $A \subset B$ with $f, g : B \to B$, we have $q(B) = q(A) \cdot q(B/A)$, in the usual sense that if two are finite, so is the third, and the relation holds. *(Proof below)*

With $C = B/A$, the lemma refers to a situation

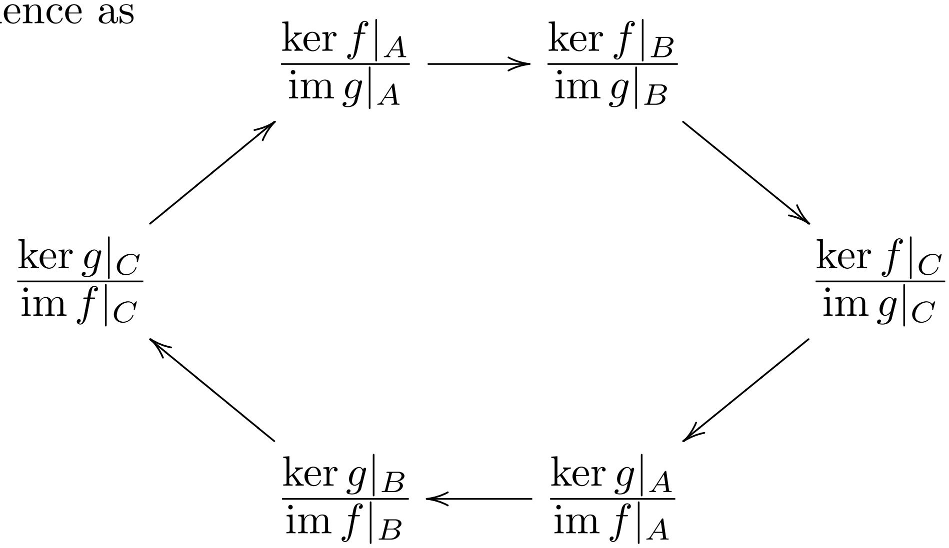
$$
\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}
$$

with columns *complexes* and rows *exact*.

A special case of the **long exact sequence in (co-) homology** will give a periodic long exact sequence

$$\ldots \to \frac{\ker f_A}{\operatorname{im} g_A} \to \frac{\ker f_B}{\operatorname{im} g_B} \to \frac{\ker f_C}{\operatorname{im} g_C} \to \frac{\ker g_A}{\operatorname{im} f_A} \to \frac{\ker g_B}{\operatorname{im} f_B} \to \frac{\ker g_C}{\operatorname{im} f_C} \to \ldots$$

The periodicity often is emphasized by writing the long exact sequence as



The numerical assertion of the Herbrand lemma is extracted from this periodic exact sequence by *Euler-Poincaré characteristics:*

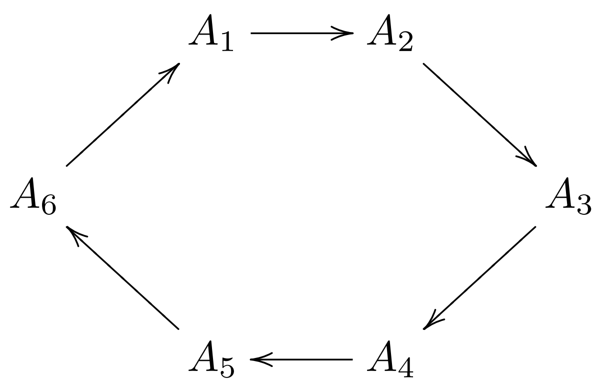
For exact sequences of *finite* abelian groups

$$0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_{n-1} \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow 0$$

we have

$$\frac{|A_1| \cdot |A_3| \cdot |A_5| \cdot \ldots}{|A_2| \cdot |A_4| \cdot |A_6| \ldots} \;=\; 1$$

and the analogous corollary: for periodic exact



we have

$$\frac{|A_1| \cdot |A_3| \cdot |A_5|}{|A_2| \cdot |A_4| \cdot |A_6|} \;=\; 1$$

In the periodic exact sequence

$$\begin{array}{ccccccc} & & \dfrac{\ker f|_A}{\operatorname{im} g|_A} & \longrightarrow & \dfrac{\ker f|_B}{\operatorname{im} g|_B} & & \\ & \nearrow & & & & \searrow & \\ \dfrac{\ker g|_C}{\operatorname{im} f|_C} & & & & & & \dfrac{\ker f|_C}{\operatorname{im} g|_C} \\ & \nwarrow & & & & \swarrow & \\ & & \dfrac{\ker g|_B}{\operatorname{im} f|_B} & \longleftarrow & \dfrac{\ker g|_A}{\operatorname{im} f|_A} & & \end{array}$$

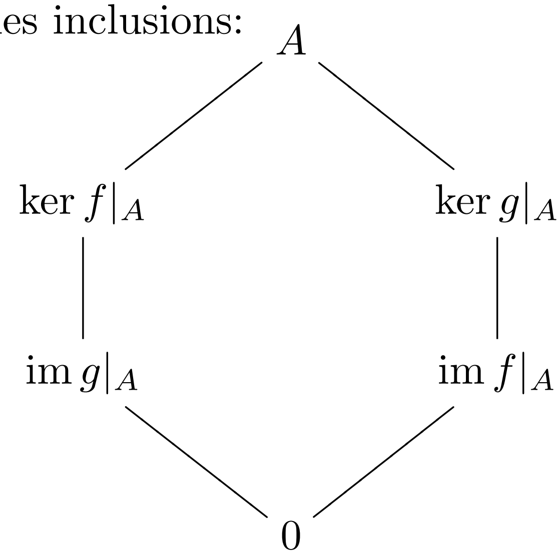
group the cardinalities belonging to A, B, C, and note the inversion for B:

$$1 \quad = \quad \frac{|A_1|}{|A_4|} \cdot \frac{|A_5|}{|A_2|} \cdot \frac{|A_3|}{|A_6|}$$

$$= \frac{[\ker f_A : \operatorname{im} g_A]}{[\ker g_A : \operatorname{im} f_A]} \cdot \frac{[\ker g_B : \operatorname{im} f_B]}{[\ker f_B : \operatorname{im} g_B]} \cdot \frac{[\ker f_C : \operatorname{im} g_C]}{[\ker g_C : \operatorname{im} f_C]} \qquad ///$$

The triviality assertion: For A finite, $\frac{[\ker f_A : \operatorname{im} g_A]}{[\ker g_A : \operatorname{im} f_A]} = 1$.

Proof: A similar but more elementary hexagonal picture is useful, with ascending lines inclusions:



By the isomorphism theorem, $A/\ker f|_A \approx \operatorname{im} f|_A$ and $A/\ker g|_A \approx \operatorname{im} g|_A$, so opposite *slanted* sides have the same indexes. By finiteness of A and multiplicativity of indices, the vertical indexes are identical. ///

A similar, useful, relatively elementary result:

Lemma: For abelian groups $A \supset B$ with a group homomorphism $f : A \to A'$, writing f_A for $f|_A$ and similarly for B,

$$[A : B] \;=\; [\ker f_A : \ker f_B] \;\cdot\; [\mathrm{im}\, f_A : \mathrm{im}\, f_B]$$

in the sense that if two of the indices are *finite*, then the third is, also, and equality holds

Proof: Certainly $A \supset \ker f_A + B \supset B$, and

$$[A : B] = [A : \ker f_A + B] \cdot [\ker f_A + B : B]$$

By isomorphism theorems,

$$\frac{A}{\ker f_A + B} \;\approx\; \frac{\mathrm{im}\, f_A}{\mathrm{im}\, f_B}$$

and

$$\frac{\ker f_A + B}{B} \;\approx\; \frac{\ker f_A}{\ker f_A \cap B} \;=\; \frac{\ker f_A}{\ker f_B} \qquad ///$$

Long exact sequence in homology: Attached to a short exact sequence of (vertical) *complexes*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow {\scriptstyle f_{i-1}} & & \downarrow {\scriptstyle g_{i-1}} & & \downarrow {\scriptstyle h_{i-1}} & & \\
0 & \longrightarrow & A_i & \longrightarrow & B_i & \longrightarrow & C_i & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle f_i} & & \downarrow {\scriptstyle g_i} & & \downarrow {\scriptstyle h_i} & & \\
0 & \longrightarrow & A_{i+1} & \longrightarrow & B_{i+1} & \longrightarrow & C_{i+1} & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle f_{i+1}} & & \downarrow {\scriptstyle g_{i+1}} & & \downarrow {\scriptstyle h_{i+1}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

the Snake Lemma gives a long exact sequence involving (co-) homology quotients $\ker f_i/\mathrm{im}\, f_{i-1}$, $\ker g_i/\mathrm{im}\, g_{i-1}$, $\ker h_i/\mathrm{im}\, h_{i-1}$

namely,

$$\ldots \longrightarrow \frac{\ker f_i}{\operatorname{im} f_{i-1}} \longrightarrow \frac{\ker g_i}{\operatorname{im} g_{i-1}} \longrightarrow \frac{\ker h_i}{\operatorname{im} h_{i-1}}$$

$$\longrightarrow \frac{\ker f_{i+1}}{\operatorname{im} f_i} \longrightarrow \frac{\ker g_{i+1}}{\operatorname{im} g_i} \longrightarrow \frac{\ker h_{i+1}}{\operatorname{im} h_i} \longrightarrow \ldots$$

That is, a short exact sequence of complexes gives a long exact sequence of (co-) homology groups of the complexes.

The bare definition of the Herbrand quotient involves a short exact sequence of *periodic* complexes.

A blunt question: *where do complexes come from?*

Still avoiding a general discussion of origins of complexes and short exact sequences of complexes, Hilbert's Theorem 90 suggests a (non-topological) source: with finite cyclic $G = \langle\sigma\rangle$, with $t = \sum_{g\in G} g$, and an exact sequence $0 \to A \to B \to C \to 0$ of G-modules, we get a short exact sequence of complexes as in the Herbrand quotient situation:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow t & & \downarrow t & & \downarrow t & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow \sigma-1 & & \downarrow \sigma-1 & & \downarrow \sigma-1 & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow t & & \downarrow t & & \downarrow t & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow \sigma-1 & & \downarrow \sigma-1 & & \downarrow \sigma-1 & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

and the Herbrand quotient lemma gives

$$\frac{[\ker t|_A : \mathrm{im}\,(\sigma-1)|_A]}{[\ker(\sigma-1)|_A : \mathrm{im}\,t|_A]} \times \frac{[\ker(\sigma-1)|_B : \mathrm{im}\,t|_B]}{[\ker t|_B : \mathrm{im}\,(\sigma-1)|_B]}$$

$$\times \frac{[\ker t|_C : \mathrm{im}\,(\sigma-1)|_C]}{[\ker(\sigma-1)|_C : \mathrm{im}\,t|_C]} \quad = \quad 1$$

for an exact sequence $0 \to A \to B \to C \to 0$ of modules for a finite cyclic group $G = \langle \sigma \rangle$.

Local cyclic norm index theorem: (Also, see Lang, p. 187 ff.) For a cyclic extension K/k of degree n of local fields, with Galois group $G = \langle \sigma \rangle$ and ramification index e, integers $\mathfrak{o} \subset k$ and $\mathfrak{O} \subset K$,

$$[k^\times : N_k^K K^\times] \;=\; n \qquad\qquad [\mathfrak{o}^\times : N_k^K \mathfrak{O}^\times] \;=\; e$$