- Classfield Theory...
- Herbrand quotient as Euler-Poincaré characteristic
- Elementary kernel-image relations
- Long exact sequences from short exact of complexes
- Norm index equality for cyclic extensions of local fields

We can produce long exact sequences in (co-) homology from short exact sequences of *complexes*, but have no *general* mechanism to produce short exact sequences of complexes.

Nevertheless, an obvious example is the short exact sequence of complexes produced from a short exact sequence $0 \to A \to B \to C \to 0$ of G-modules for finite cyclic G, by the universal complex construction

$$A \longrightarrow \left(\cdots \xrightarrow{t} A \xrightarrow{\sigma-1} A \xrightarrow{t} A \xrightarrow{\sigma-1} \cdots \right)$$

for any G-module A, where $G = \langle \sigma \rangle$ and $t = \sum_{g \in G} g$.

Herbrand quotients: A periodic *complex*

$$\cdots \xrightarrow{f} A \xrightarrow{g} A \xrightarrow{f} A \xrightarrow{g} \cdots$$

has just two (co-) homology groups,

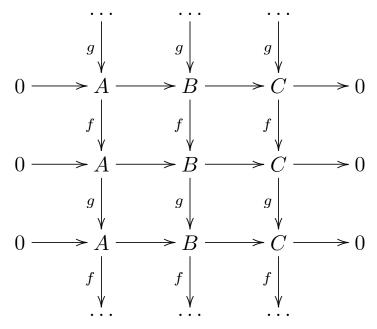
$$\frac{\ker f|_A}{\operatorname{im} g_A} \qquad \frac{\ker g|_A}{\operatorname{im} f_A}$$

and the Herbrand quotient is of A, f, g is

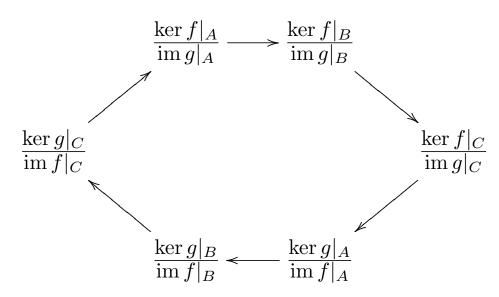
$$q_{f,g}(A) = \frac{[\ker f : \operatorname{im} g]}{[\ker g : \operatorname{im} f]}$$

Key Lemma: For finite A, q(A) = 1. For f-stable, g-stable subgroup $A \subset B$ with $f, g: B \to B$, we have $q(B) = q(A) \cdot q(B/A)$, in the usual sense that if two are finite, so is the third, and the relation holds.

With C=B/A, the lemma refers to a short exact sequence of complexes



A special case of the long exact sequence in (co-) homology will give a periodic long exact sequence



By Euler-Poincaré characteristics:

$$1 = \frac{\left[\ker f_A : \operatorname{im} g_A\right]}{\left[\ker g_A : \operatorname{im} f_A\right]} \cdot \frac{\left[\ker g_B : \operatorname{im} f_B\right]}{\left[\ker f_B : \operatorname{im} g_B\right]} \cdot \frac{\left[\ker f_C : \operatorname{im} g_C\right]}{\left[\ker g_C : \operatorname{im} f_C\right]}$$
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The triviality assertion: For A finite,
$$\frac{[\ker f_A : \operatorname{im} g_A]}{[\ker g_A : \operatorname{im} f_A]} = 1.$$
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Another useful, relatively elementary result:

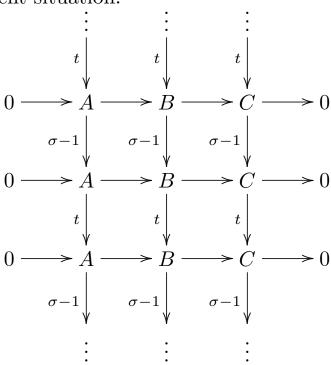
Lemma: For abelian groups $A \supset B$ with a group homomorphism $f: A \to A'$, writing f_A for $f|_A$ and similarly for B,

$$[A:B] = [\ker f_A : \ker f_B] \cdot [\operatorname{im} f_A : \operatorname{im} f_B]$$

in the sense that if two of the indices are *finite*, then the third is, also, and equality holds.

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Avoiding a general discussion of origins of complexes, Hilbert's Theorem 90 suggests a (non-topological) source: with finite cyclic $G = \langle \sigma \rangle$, with $t = \sum_{g \in G} g$, an exact sequence $0 \to A \to B \to C \to 0$ of G-modules gives a short exact sequence of complexes as in the Herbrand quotient situation:



and the Herbrand quotient lemma gives

$$\frac{[\ker t|_A : \operatorname{im}(\sigma - 1)|_A]}{[\ker(\sigma - 1)|_A : \operatorname{im}t|_A]} \times \frac{[\ker(\sigma - 1)|_B : \operatorname{im}t|_B]}{[\ker t|_B : \operatorname{im}(\sigma - 1)|_B]} \times \frac{[\ker t|_C : \operatorname{im}(\sigma - 1)|_C]}{[\ker(\sigma - 1)|_C : \operatorname{im}t|_C]} = 1$$

for an exact sequence $0 \to A \to B \to C \to 0$ of modules for a finite cyclic group $G = \langle \sigma \rangle$.

Local cyclic norm index theorem: (Also, see Lang, p. 187 ff.) For a cyclic extension K/k of degree n of local fields, with Galois group $G = \langle \sigma \rangle$ and ramification index e, integers $\mathfrak{o} \subset k$ and $\mathfrak{O} \subset K$,

$$[k^{\times}:N_k^KK^{\times}] = n \qquad [\mathfrak{o}^{\times}:N_k^K\mathfrak{O}^{\times}] = e$$

Proof: Apply the Herbrand quotient lemma to

$$0 \longrightarrow \mathfrak{D}^{\times} \longrightarrow K^{\times} \longrightarrow \mathbb{Z} \longrightarrow 0$$

where ord : $K^{\times} \to \mathbb{Z}$. Since Galois preserves $|\cdot|_K$, the action of G on this copy of \mathbb{Z} is *trivial*. Thus,

$$\ker t|_{\mathbb{Z}} = \{0\} \qquad \operatorname{im} t|_{\mathbb{Z}} = n \cdot \mathbb{Z}$$

and

$$\ker(\sigma-1)|_{\mathbb{Z}} = \mathbb{Z}$$
 $\operatorname{im}(\sigma-1)|_{\mathbb{Z}} = 0$

SO

$$q_{\sigma-1,t}(\mathbb{Z}) = \frac{[\ker(\sigma-1)|_{\mathbb{Z}} : \operatorname{im} t|_{\mathbb{Z}}]}{[\ker t|_{\mathbb{Z}} : \operatorname{im} (\sigma-1)|_{\mathbb{Z}}]} = \frac{[\mathbb{Z} : n \cdot \mathbb{Z}]}{[\{0\} : \{0\}]} = n$$

Theorem 90 is $\ker t|_{K^{\times}} = \operatorname{im} (\sigma - 1)|_{K^{\times}}$. Thus,

$$q_{\sigma-1,t}(K^{\times}) = \frac{[\ker(\sigma-1)|_{K^{\times}} : \operatorname{im} t|_{K^{\times}}]}{[\ker t|_{K^{\times}} : \operatorname{im} (\sigma-1)|_{K^{\times}}]}$$

$$= \frac{[k^{\times}: N_k^K K^{\times}]}{1} = [k^{\times}: N_k^K K^{\times}]$$

Thus, the Herbrand Lemma gives

$$n = q_{\sigma-1,t}(\mathbb{Z}) = \frac{q_{\sigma-1,t}(K^{\times})}{q_{\sigma-1,t}(\mathfrak{O}^{\times})} = \frac{[k^{\times} : N_k^K K^{\times}]}{q_{\sigma-1,t}(\mathfrak{O}^{\times})}$$

We show that $q_{\sigma-1,t}(\mathfrak{O}^{\times})=1$.

There is a normal basis x_1, \ldots, x_n for K/k, that is, G acts transitively on the x_i . Multiply every x_i by the same sufficiently high power of a local parameter in k to preserve the normal basis property, and to put all the x_i inside a sufficiently high power of the maximal ideal in \mathfrak{O} such that the exponential map is defined on $V = \sum_i \mathfrak{o} x_i$, and inverse given by logarithm is defined on its image $U = \exp V$.

Claim: $\ker(\sigma - 1)|_V = \operatorname{im} t|_V$ and $\ker t|_V = \operatorname{im} (\sigma - 1)|_V$.

Proof of claim: With $(\sigma-1)\sum_i c_i x_i = 0$ with $c_i \in k$, all coefficients are the same, by transitivity. On the other hand, if the coefficients are the same, certainly the element is in $\ker(\sigma-1)$. Application of t to c_1x_1 produces all elements with identical coefficients. Thus, $\ker(\sigma-1)|_V = \operatorname{im} t|_V$. This is one equality.

For the other equality, index so that $x_i = \sigma^{i-1}(x_1)$.

Vanishing $t \cdot \sum_i c_i x_i = 0$ is exactly $(\sum_i c_i)(\sum_i x_i) = 0$, which is $\sum_i c_i = 0$. Then

$$\sum_{i} c_{i} x_{i} = \sum_{i} c_{i} (x_{i} - x_{1}) = \sum_{i} c_{i} (\sigma^{i-1} - 1) x_{1} \in (\sigma - 1) \sum_{i} \mathfrak{o} x_{i}$$

so
$$\ker t|_V = \operatorname{im}(\sigma - 1)|_V$$
. ///

Since the Galois action is *continuous* on K, it commutes with exp and log where the series converge. Thus, $V = \exp U$ is a G-module with the same Herbrand-related quotients as U, namely

$$\ker(\sigma - 1)|_U = \operatorname{im} t|_U$$
 and $\ker t|_U = \operatorname{im} (\sigma - 1)|_U$

Since $[\mathfrak{O}^{\times}: U] < \infty$, by the Lemma $q_{\sigma-1,t}(\mathfrak{O}^{\times}/U) = 1$, and, again by the Herbrand Lemma,

$$1 = q_{\sigma-1,t}(V) = q_{\sigma-1,t}(U) = \frac{q_{\sigma-1,t}(\mathfrak{O}^{\times})}{q_{\sigma-1,t}(\mathfrak{O}^{\times}/U)} = q_{\sigma-1,t}(\mathfrak{O}^{\times})$$

From this,

$$1 = q_{\sigma-1,t}(\mathfrak{O}^{\times}) = \frac{[\mathfrak{o}^{\times} : N_k^K \mathfrak{O}^{\times}]}{[\ker t|_{\mathfrak{O}^{\times}} : \operatorname{im}(\sigma-1)|_{\mathfrak{O}^{\times}}]}$$

Since $|\sigma x/x| = 1$, by Theorem 90, $\ker t|_{\mathfrak{O}^{\times}} = \operatorname{im}(\sigma - 1)|_{K^{\times}}$. Thus,

$$[\ker t|_{\mathfrak{O}^{\times}} : \operatorname{im}(\sigma - 1)|_{\mathfrak{O}^{\times}}] = [\operatorname{im}(\sigma - 1)|_{K^{\times}} : \operatorname{im}(\sigma - 1)|_{\mathfrak{O}^{\times}}]$$
$$= [\operatorname{im}(\sigma - 1)|_{K^{\times}} : \operatorname{im}(\sigma - 1)|_{k^{\times}\mathfrak{O}^{\times}}]$$

Using $[A:B] = [\ker f|_A : \ker f|_B] \cdot [\operatorname{im} f|_A : \operatorname{im} f|_B]$ for $A \supset B$, this is $\frac{[K^{\times} : k^{\times} \mathfrak{D}^{\times}]}{[\ker(\sigma - 1)|_{K^{\times}} : \ker(\sigma - 1)|_{k^{\times} \mathfrak{D}^{\times}}]}$

Essentially by definition, $[K^{\times}: k^{\times}\mathfrak{O}^{\times}] = e$, and

$$[\ker(\sigma-1)|_{K^{\times}}:\ker(\sigma-1)|_{k^{\times}\mathfrak{I}^{\times}}] = [k^{\times}:k^{\times}] = 1$$
 so
$$[\ker t|_{\mathfrak{I}^{\times}}:\operatorname{im}(\sigma-1)|_{\mathfrak{I}^{\times}}] = e.$$

Thus,

$$1 = \frac{\left[\mathfrak{o}^{\times} : N_{k}^{K}\mathfrak{O}^{\times}\right]}{\left[\ker t|_{\mathfrak{O}^{\times}} : \operatorname{im}\left(\sigma - 1\right)|_{\mathfrak{O}^{\times}}\right]} = \frac{\left[\mathfrak{o}^{\times} : N_{k}^{K}\mathfrak{O}^{\times}\right]}{e}$$

and the cyclic local norm index theorem is done.

Elementary abelian group theory and induction give

Corollary: For finite abelian extension K/k of local fields,

$$[k^{\times}: N_k^K K^{\times}] \leq [K:k]$$
 and $[\mathfrak{o}^{\times}: N_k^K \mathfrak{O}^{\times}] \leq e$ ///

Remark: Local classfield theory asserts *equalities* here for all finite abelian extensions, not only cyclic.