Continuing to review the simple case (haha!) of number theory over $\mathbb{Z}$:

Another example of the possibly-suprising application of othe things to number theory.

Riemann's explicit formula

More interesting than a Prime Number Theorem is the *precise* relationship between primes and zeros of zeta found by Riemann.

The idea applies to any zeta or L-function for which we know an analytic continuation and other reasonable properties.

It took 40 years for [Hadamard 1893], [vonMangoldt 1895], and others to complete Riemann's 1857-8 sketch of the *Explicit Formula* relating primes to zeros of the Euler-Riemann zeta function. Even then, lacking a zero-free strip inside the critical strip, the Explicit Formula does *not* yield a Prime Number Theorem, despite giving a precise relationship between primes and zeros of zeta.

Riemann's explicit formula

Riemann's dramatic relation between primes and zeros of the zeta function depends on many ideas undeveloped in Riemann's time. Thus, the following sketch, roughly following Riemann, is not a proof. Rather, the sketch tells which supporting ideas need development to produce a proof.

that $\zeta(s)$ has both its *Euler product* expansion in a half-plane

$$\zeta(s) \;=\; \sum_{n\geq 1} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \frac{1}{1-\frac{1}{p^s}} \qquad\qquad (\mathrm{Re}\, s > 1)$$

Riemann already knew $\zeta(s)$ has a *meromorphic continuation* throughout $\mathbb{C}$ (see below).

If we believe, as Riemann did, and as Hadamard and others later *proved*, that is *also* has a *Riemann-Hadamard product* expansion

$$(s-1)\,\zeta(s)$$

$$= \; e^{a+bs} \prod_{\rho} \Big(1 - \frac{s}{\rho}\Big)\, e^{s/\rho} \cdot \prod_{n=1}^{\infty} \Big(1 + \frac{s}{2n}\Big)\, e^{-s/2n}$$

product over ρ non-trivial zero of ζ, for all $s \in \mathbb{C}$.

Then, following Riemann, extract tangible information from the equality of the two products

$$(s-1)\, \prod_{p} \frac{1}{1 - \frac{1}{p^s}}$$

$$= \quad e^{a+bs} \prod_{\rho} \Big(1 - \frac{s}{\rho}\Big)\, e^{s/\rho} \cdot \prod_{n=1}^{\infty} \Big(1 + \frac{s}{2n}\Big)\, e^{-s/2n} \qquad (\mathrm{Re}\, s > 1)$$

First, take logarithmic derivatives of both sides, using $-\log(1-x) = x + x^2/2 + x^3/3 + \ldots$ on the left-hand side:

$$\frac{1}{s-1} - \sum_{m\geq 1,\ p} \frac{\log p}{p^{ms}}$$

$$= \ b \ + \ \sum_\rho \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) \ + \ \sum_n \Big(\frac{1}{s+2n} - \frac{1}{2n}\Big)$$

A slight rearrangement:

$$\sum_{m\geq 1,\ p} \frac{\log p}{p^{ms}}$$

$$= \ \frac{1}{s-1} - b \ - \ \sum_\rho \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) \ - \ \sum_n \Big(\frac{1}{s+2n} - \frac{1}{2n}\Big)$$

The left-hand side needs $\mathrm{Re}\, s > 1$ for convergence, while the right-hand side converges for all $s \in \mathbb{C}$ apart from the visible poles at 1, the non-trivial zeros ρ, and the trivial zeros $2, 4, 6, \ldots$.

Next, apply the *Perron identity*

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{Y^s}{s}\,ds \;=\; \begin{cases} 1 & (\text{for } Y > 1) \\ 0 & (\text{for } 0 < Y < 1) \end{cases} \qquad (\text{for } \sigma > 0)$$

Really, we have to be slightly careful:

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{Y^s}{s}\,ds \;=\; \begin{cases} 1 & (\text{for } Y > 1) \\ 0 & (\text{for } 0 < Y < 1) \end{cases} \qquad (\text{for } \sigma > 0)$$

If we can apply this to entire expressions, by

$$f \;\longrightarrow\; \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} f(s)\cdot \frac{X^s}{s}\,ds \qquad (\text{with } \sigma > 1)$$

term-wise to the left-hand side, *and* use residues *term*-wise to evaluate the right-hand side, we would have

$$\sum_{p^m < X} \log p \;=\; (X-1) \;-\; b$$

$$-\;\sum_\rho \Big(\frac{X^\rho}{\rho} + \frac{1}{-\rho} + \frac{1}{\rho}\Big) \;-\; \sum_n \Big(\frac{X^{-2n}}{-2n} + \frac{1}{2n} - \frac{1}{2n}\Big)$$

which simplifies to von Mangoldt's reformulation of **Riemann's Explicit Formula:**

$$\sum_{p^m < X} \log p \;=\; X \;-\; (b+1) \;-\; \sum_{\rho} \frac{X^\rho}{\rho} \;+\; \sum_{n \geq 1} \frac{X^{-2n}}{2n}$$

Slightly more precisely, because of the way the Perron integral transform is applied, and the fragility of the convergence,

$$\sum_{p^m < X} \log p \;\;=\;\; X \;-\; (b+1) \;-\; \lim_{T \to \infty} \sum_{|\mathrm{Im}\,(\rho)| < T} \frac{X^\rho}{\rho} \;+\; \sum_{n \geq 1} \frac{X^{-2n}}{2n}$$

The Riemann-Hadamard product needs both *generalities* about Weierstraß-Hadamard product expressions for entire functions of prescribed growth, and *specifics* about the growth of the *analytic continuation* of $\zeta(s)$.

For future reference:

The two sides of the equality of logarithmic derivatives are very different.

The logarithmic derivative of the Euler product converges well in right half-planes, and converges all the better farther to the right.

The logarithmic derivative of the Riemann-Hadamard product does not converge powerfully, but is not restricted to a half-plane, and its poles are exhibited explicitly by the expression.

Analytic continuation and functional equation of $\zeta(s)$

The following ideas gained publicity and importance from Riemann's 1857-8 paper, but were apparently known before Riemann's time.

The key is that the completed zeta function has an *integral representation* in terms of an *automorphic form*, namely, the simplest theta function. Both the analytic continuation and the functional equation of zeta follow from this integral representation using a parallel functional equation of the theta function, the latter demonstrated by *Poisson summation*.

Elementary-but-doomed argument It is worthwhile to see that simple calculus can extend the domain of $\zeta(s)$ a little. The idea is to pay attention to *quantitative* aspects of the integral test. That is,

$$\zeta(s) - \frac{1}{s-1} \;=\; \zeta(s) - \int_1^\infty \frac{dx}{x^s}$$

$$= \sum_n \Big(\frac{1}{n^s} - \int_n^{n+1} \frac{dx}{x^s}\Big)$$

$$= \sum_n \Big(\frac{1}{n^s} - \frac{1}{s-1}\Big[\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\Big]\Big)$$

Even for complex s, we have a Taylor-Maclaurin expansion with error term

$$(n+1)^{1-s} \;=\; \big(n\cdot(1+\frac{1}{n})\big)^{1-s} \;=\; n^{1-s}\cdot\big(1+\frac{1-s}{n}+O(\tfrac{1}{n^2})\big)$$

$$= \frac{1}{n^{s-1}} - \frac{s-1}{n^s} + O(\frac{s-1}{n^{s+1}})$$

The constant in the big-O term is *uniform* in n for fixed s. Thus,

$$\frac{1}{n^s} - \frac{1}{s-1}\Big[\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\Big] \;=\; O\big(\tfrac{1}{n^{s+1}}\big)$$

That is, for fixed $\mathrm{Re}(s) > 0$, we have *absolute convergence* of

$$\sum_n \Big(\frac{1}{n^s} - \frac{1}{s-1}\Big[\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\Big]\Big)$$

in the larger region $\mathrm{Re}(s) > 0$.

A similar but increasingly complicated device produces a meromorphic continuation to half-planes $\mathrm{Re}(s) > \ell$. However, this approach is under-powered...

A more serious argument Euler's integral for the **gamma function** is

$$\Gamma(s) \;=\; \int_0^\infty e^{-t}\, t^s \,\frac{dt}{t}$$

Among other roles, the gamma function $\Gamma(s)$ interpolates the *factorial function*: integration by parts yields $\Gamma(n) \;=\; (n-1)!$ for positive integer n.

Theorem The **completed** zeta function

$$\xi(s) \;=\; \pi^{-\frac{s}{2}}\,\Gamma(\frac{s}{2})\,\zeta(s)$$

has an analytic continuation to $s \in \mathbb{C}$, except for simple poles at $s = 0, 1$, and has the *functional equation*

$$\xi(1-s) \;=\; \xi(s)$$

... and (anticipating the Riemann-Hadamard product issues) $s(s-1)\xi(s)$ is entire and **bounded in vertical strips**.

The following proof-sketch is itself an archetype.

The simplest *theta function* is

$$\theta(z) \;=\; \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$$

with z in the complex upper half-plane $\mathfrak{H}$. By Riemann's time, Jacobi's functional equation of $\theta(z)$ was well-known, as the simplest example of a larger technical phenomenon:

$$\theta(z) \;=\; \frac{1}{\sqrt{-iz}} \cdot \theta(-1/z)$$

(Proven below.) The modified version

$$\frac{\theta(iy)-1}{2} \;=\; \sum_{n=1}^{\infty} e^{-\pi n^2 y}$$

appears just below.

The connection to $\zeta(s)$ is the *integral presentation*:

Claim: For $\mathrm{Re}(s) > 1$

$$\pi^{-s/2}\,\Gamma(\frac{s}{2})\cdot\zeta(s) \;=\; \int_0^\infty \frac{\theta(iy)-1}{2}\cdot y^{s/2}\cdot\frac{dy}{y}$$

Meaning? An integral against t^s with dt/t, a *Mellin transform*, is just a *Fourier transform* in different coordinates.

Starting from the integral, for $\mathrm{Re}(s) > 1$, compute directly

$$\int_0^\infty \frac{\theta(iy)-1}{2}\,y^{s/2}\,\frac{dy}{y} \;=\; \int_0^\infty \sum_{n\geq 1} e^{-\pi n^2 y}\,y^{s/2}\,\frac{dy}{y}$$

$$=\; \sum_{n\geq 1}\int_0^\infty e^{-\pi n^2 y}\,y^{s/2}\,\frac{dy}{y} \;=\; \pi^{-s/2}\sum_{n\geq 1}\frac{1}{n^{2s}}\int_0^\infty e^{-y}\,y^{s/2}\,\frac{dy}{y}$$

by replacing y by $y/(\pi n^2)$, and this is

$$=\; \pi^{-s/2}\Gamma(\frac{s}{2})\cdot\sum_{n\geq 1}\frac{1}{n^s} \;=\; \xi(s) \qquad\qquad (\text{for } \mathrm{Re}(s) > 1)$$

Bibliography

[Edwards 1974] H.M. Edwards, *Riemann's zeta function*, Academic Press, New York, 1974.

[Hadamard 1893] J. Hadamard, *Étude sur les Propriétés des Fonctions Entières et en Particulier d'une Fonction Considérée par Riemann*, J. Math. Pures Appl. **p**, 171-215.

[Ivić 1985] A. Ivić, *The Riemann zeta function*, J. Wiley, New York, 1985.

[Iwaniec 2002] H. Iwaniec, *Spectral Methods of Automorphic Forms*, 2nd edition, AMS, Providence, 2002. [First edition, Revisto Mathematica Iberoamericana, 1995.]

[vonMangoldt 1895] H. von Mangoldt, *Zu Riemann's Abhandlung 'Über die Anzahl der Primzahlen unter einer gegebenen Grösse'*, J. Reine Angew. Math. **114**, 255-305.

[Patterson 1988] S. J. Patterson, *An introduction to the theory of the Riemann zeta function*, Cambridge University Press, 1988.

[Titchmarsh 1951] E.C. Titchmarsh, *The theory of the Riemann zeta function*, Oxford University Press, 1951.

[Weil 1952] A. Weil, *Sur les 'formules explicites' de la théorie des nombres premiers*, Comm. Lund (vol. dedié à Marcel Riesz) 252-265. (*Oeuvres sci.* [1952b], Vol. II, Springer, New York, 1979.)

[Weil 1972] A. Weil, *Sur les formules explicites de la théorie des nombres*, Izv. Akad. Nauk. (ser. Math.) **36** (1972), 3-18. (*Oeuvres sci.* [1972], Vol. III, Springer, New York, 1979.)