Dedekind zeta functions, class number formulas, ...

$$\zeta_k(s) \;=\; \sum_{0 \neq \mathfrak{a} \subset \mathfrak{o}} \frac{1}{N\mathfrak{a}^s} \;=\; \prod_{\mathfrak{p} \text{ prime in } \mathfrak{o}} \frac{1}{1 - N\mathfrak{p}^{-s}}$$

The Euler product and sum expressions for $\zeta_k(s)$ converge absolutely for $\mathrm{Re}(s) > 1$. [Previously.]

The simplest family of rings of algebraic integers typically *not* PIDs, but with the simple feature of *finitely-many units*, is *complex quadratic* $k = \mathbb{Q}(\sqrt{-D})$ for $D > 0$. Let the ring of algebraic integers be $\mathfrak{o}$, quadratic symbol $\chi(p) = (-D/p)_2$, N the conductor of χ, $h(\mathfrak{o})$ the *class number*. Then

$$h(\mathfrak{o}) \;=\; \frac{-i \cdot |\mathfrak{o}^\times| \cdot \mathrm{coarea}(\mathfrak{o})}{\sum_a \chi(a) e^{2\pi i a/N}} \sum_{a \bmod N} \big(\frac{a}{N} - \tfrac{1}{2}\big) \cdot \chi(a)$$

Discussion and elaboration ...

Again, ...

$$\mathfrak{o} \;=\; \begin{cases} \mathbb{Z}[\sqrt{-D}] & = \quad \mathbb{Z} \oplus \mathbb{Z}\sqrt{-D} \qquad (\text{for } -D = 2, 3 \bmod 4) \\ \mathbb{Z}[\frac{1+\sqrt{-D}}{2}] & = \quad \mathbb{Z} \oplus \mathbb{Z}\frac{1+\sqrt{-D}}{2} \quad (\text{for } -D = 1 \bmod 4) \end{cases}$$

$\mathfrak{o}$ is a free $\mathbb{Z}$-module of rank 2, and is a *lattice* in $\mathbb{C}$: $\mathfrak{o}$ is a *discrete* subgroup of $\mathbb{C}$, and $\mathbb{C}/\mathfrak{o}$ is *compact*.

Galois norm is the *complex* norm-squared: $N^k_{\mathbb{Q}}(\alpha) = \alpha \cdot \bar{\alpha} = |\alpha|^2$.

Lemma: For a lattice Λ in $\mathbb{C}$, the Epstein zeta function

$$Z_\Lambda(s) \;=\; \sum_{0 \neq \lambda \in \Lambda} \frac{1}{|\lambda|^{2s}}$$

has a meromorphic continuation to $\mathrm{Re}(s) > \frac{1}{2}$ and

$$Z_\Lambda(s) \;=\; \frac{\pi}{\text{co-area}\,\Lambda} \cdot \frac{1}{s-1} \;+\; (\text{holomorphic near } s = 1)$$

[Last time.]

Corollary: For complex quadratic k,

$$\zeta_k(s) \;=\; \sum_{[\mathfrak{b}]} \sum_{\mathfrak{a}\sim\mathfrak{b}} \frac{1}{N\mathfrak{a}^s} \quad\sim\quad \frac{\pi\cdot h(\mathfrak{o})}{|\mathfrak{o}^\times|\cdot \text{coarea}(\mathfrak{o})(s-1)} + (\text{holo at } s=1)$$

Proof: [Last time:] co-area$(\mathfrak{b}^{-1}) = N\mathfrak{b}^{-1}\cdot$ coarea$(\mathfrak{o})$. ///

Corollary With $\chi(p) = (-D/p)_2$,

$$\frac{\pi\cdot h(\mathfrak{o})}{|\mathfrak{o}^\times|\cdot \text{coarea}(\mathfrak{o})} \;=\; L(1,\chi)$$

Proof: [Last time:] From the *factorization*

$$\zeta_k(s) \;=\; \zeta_{\mathbb{Q}}(s)\cdot L(s,\chi)$$

Since $\zeta(s) = \zeta_{\mathbb{Q}}(s)$ has residue 1 at $s=1$, the value $L(1,\chi)$ is the residue of $\zeta_k(s)$ at $s=1$. ///

For complex quadratic k, the special value $L(1,\chi)$ has a finite, closed-form expression. Recall that the *conductor* N of χ is a positive integer such that $\chi(p)$ depends only on p mod N.

Claim: The *conductor* N of $\chi(p) = (-D/p)_2$ is

$$N = \begin{cases} D & (\text{for } -D = 1 \bmod 4) \\ \\ 4D & (\text{for } -D = 2, 3 \bmod 4) \end{cases}$$

Proof: Use quadratic reciprocity. For D an odd *prime*,

$$\left(\frac{-D}{p}\right)_2 = \left(\frac{-1}{p}\right)_2 \left(\frac{D}{p}\right)_2 = (-1)^{\frac{p-1}{2}} \cdot (-1)^{\frac{p-1}{2}\frac{D-1}{2}} \left(\frac{p}{D}\right)_2$$

$$= (-1)^{\frac{p-1}{2}\frac{D+1}{2}} \left(\frac{p}{D}\right)_2 = \begin{cases} \left(\frac{p}{D}\right)_2 & (\text{for } D = 3 \bmod 4) \\ \\ (-1)^{\frac{p-1}{2}} \cdot \left(\frac{p}{D}\right)_2 & (\text{for } D = 1 \bmod 4) \end{cases}$$

For $D = 2q$ with odd prime q,

$$\left(\frac{-D}{p}\right)_2 = \left(\frac{-1}{p}\right)_2 \left(\frac{2}{p}\right)_2 \left(\frac{q}{p}\right)_2 = (-1)^{\frac{p-1}{2}} (-1)^{\frac{p^2-1}{8}} (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{p}{q}\right)_2$$

Here, as $(-1)^{(p^2-1)/8}$ is a slightly un-transparent interpolation of the quadratic symbol for 2, we must check the cases $p = 1, 3, 5, 7 \bmod 8$ to see that, no matter the congruence class of q, the aggregate is only defined mod $8q = 4(2q)$.

For $D = q_1 \ldots q_\ell$ with odd primes q_j,

$$\left(\frac{-D}{p}\right)_2 = (-1)^{\frac{p-1}{2}[1+\frac{q_1-1}{2}+\ldots+\frac{q_\ell-1}{2}]} \left(\frac{p}{q_1}\right)_2 \cdots \left(\frac{p}{q_\ell}\right)_2$$

With ν the number of $q_j = 3 \bmod 4$, the power of -1 is $(-1)^{\frac{p-1}{2}(1+\nu)}$. For $\nu = 1 \bmod 4$, this depends on $p \bmod 4$, and $q_1 \ldots q_\ell = 3 \bmod 4$, while for $\nu = 3 \bmod 4$ this is $+1$, and $q_1 \ldots q_\ell = 1 \bmod 4$. A similar consideration applies to $D = 2q_1 \ldots q_\ell$. ///

Remark: The precise determination of the conductor of χ for quadratic characters χ accounts for a classical usage: for square-free integer d,

$$\text{discriminant } \mathbb{Q}(\sqrt{d}) \;=\; \begin{cases} |d| & (\text{for } d = 1 \bmod 4) \\ \\ 4|d| & (\text{for } d = 2, 3 \bmod 4) \end{cases}$$

$$= \;\text{conductor of } \left(\frac{d}{*}\right)_2$$

This appears to differ from the *square* of *co-area* of $\mathfrak{o}$ by a factor of 4: for example,

$$\text{co-area } \mathbb{Z}[\sqrt{-5}] \;=\; \text{area of rectangle spanned by } 1, \sqrt{-5} \;=\; \sqrt{5}$$

while the discriminant/conductor is 20. Later, we will find that the best normalization of measure on $\mathbb{C}$ rectifies this!

The Fourier expansion of the sawtooth function is

$$s(x) \;=\; x - \tfrac{1}{2} \;=\; \frac{-1}{2\pi i}\sum_{n\neq 0}\frac{e^{2\pi i n x}}{n} \qquad\qquad \text{(for } 0 < x < 1\text{)}$$

The standard discussion of the Dirichlet kernel [for example, see *Functions on Circles* in http://.../˜garrett/m/mfms, course notes from 2005-6] shows that Fourier series of piecewise differentiable functions f with left and right limits at discontinuities *do* converge, and to f, at points where f is differentiable.

Thus,

$$\sum_{a \bmod N} \chi(a)\cdot\left(\frac{a}{N} - \tfrac{1}{2}\right) \;=\; \sum_{a \bmod N} \chi(a)\cdot s\left(\frac{a}{N}\right)$$

$$= \;\frac{-1}{2\pi i}\sum_{a}\chi(a)\sum_{n\neq 0}\frac{e^{2\pi i n a/N}}{n} \;=\; \frac{-1}{2\pi i}\sum_{n\neq 0}\frac{\chi(n)}{n}\cdot\sum_{a}\chi(a)e^{2\pi i a/N}$$

by replacing a by $an^{-1} \bmod N$. Since $\chi(-1) = -1$ (!!!)...

In fact, for quadratic characters, $\chi(-1)$ does tell whether the field is *real* or *complex*:

Lemma: For quadratic characters χ,

$$\chi(-1) \;=\; \begin{cases} -1 & (\text{for } \chi(p) = (-D/p)_2) \\ \\ +1 & (\text{for } \chi(p) = (D/p)_2) \end{cases} \qquad\qquad (\text{squarefree } D > 0)$$

Proof: As a simple case, take D odd *prime*. The conductor is either D or $4D$. For a *prime* $p = -1 \bmod 4D$,

$$\chi(-1) \;=\; \begin{pmatrix} -D \\ p \end{pmatrix}_2 \;=\; (-1)^{\frac{p-1}{2}} (-1)^{\frac{p-1}{2}\frac{D-1}{2}} \begin{pmatrix} p \\ D \end{pmatrix}_2$$

$$=\; (-1)^{\frac{p-1}{2}\frac{D+1}{2}} \begin{pmatrix} -1 \\ D \end{pmatrix}_2 = \; (-1)^{\frac{p-1}{2}\frac{D+1}{2}} (-1)^{\frac{D-1}{2}} \;=\; (-1)^{\frac{p-1}{2}\cdot D} \;=\; -1$$

since $p = 3 \bmod 4$. For $(D/p)_2$ ...

... with *prime* $p = -1 \bmod 4D$,

$$\chi(-1) \;=\; \binom{D}{p}_2 \;=\; (-1)^{\frac{p-1}{2}\,\frac{D-1}{2}} \binom{p}{D}_2$$

$$=\; (-1)^{\frac{p-1}{2}\,\frac{D-1}{2}} \binom{-1}{D}_2 = \; (-1)^{\frac{p-1}{2}\,\frac{D-1}{2}} (-1)^{\frac{D-1}{2}} \;=\; +1$$

Dirichlet's theorem on primes in arithmetic progressions gives infinitely-many primes $p = -1 \bmod 4D$, but this is excessive.

Instead, with $n = -1 \bmod 4D$, factor $n = q_1 \ldots q_\ell$, apply quadratic reciprocity, and track parities, as we did in the determination of the conductor of quadratic characters. And factor D... ///

Thus, indeed, $\chi(-1) = -1$ for complex quadratic fields. Back to the class number formula computation...

So far,

$$\sum_{a \bmod N} \chi(a) \cdot \left(\frac{a}{N} - \tfrac{1}{2}\right) \;=\; \frac{-1}{2\pi i} \sum_{n \neq 0} \frac{\chi(n)}{n} \cdot \sum_a \chi(a) e^{2\pi i a/N}$$

Since $\chi(-1) = -1$, the summands $\chi(n)/n$ for $\pm n$ are *identical*, rather than *cancelling*, so

$$\sum_{a \bmod N} \chi(a) \cdot \left(\frac{a}{N} - \tfrac{1}{2}\right) \;=\; \frac{-1}{\pi i} \cdot L(1,\chi) \cdot \sum_a \chi(a) e^{2\pi i a/N}$$

and

$$L(1,\chi) \;=\; \frac{-\pi i}{\sum_a \chi(a) e^{2\pi i a/N}} \sum_{a \bmod N} \chi(a) \cdot \left(\frac{a}{N} - \tfrac{1}{2}\right)$$

Thus,

$$\frac{\pi \cdot h(\mathfrak{o})}{|\mathfrak{o}^\times| \cdot \text{co-area}(\mathfrak{o})} \;=\; \frac{-\pi i}{\sum_a \chi(a) e^{2\pi i a/N}} \sum_{a \bmod N} \big(\frac{a}{N} - \tfrac{1}{2}\big) \cdot \chi(a)$$

and, for *complex* quadratic fields,

$$h(\mathfrak{o}) \;=\; \frac{-i \cdot |\mathfrak{o}^\times| \cdot \text{co-area}(\mathfrak{o})}{\sum_a \chi(a) e^{2\pi i a/N}} \sum_{a \bmod N} \big(\frac{a}{N} - \tfrac{1}{2}\big) \cdot \chi(a)$$

Claim:

$$\left| \frac{\text{co-area}(\mathfrak{o})}{\sum_a \chi(a) e^{2\pi i a/N}} \right| \;=\; \frac{1}{2}$$

Proof: For $-D = 1 \bmod 4$, $\mathfrak{o} = \mathbb{Z}[\frac{1+\sqrt{-D}}{2}]$, and the co-area of $\mathfrak{o}$ is

$$\det \begin{pmatrix} \text{Re}\,(1) & \text{Im}\,(1) \\ \text{Re}\,(\frac{1+\sqrt{-D}}{2}) & \text{Im}\,(\frac{1+\sqrt{-D}}{2}) \end{pmatrix} \;=\; \det \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{\sqrt{D}}{2} \end{pmatrix} \;=\; \frac{\sqrt{D}}{2}$$

For $-D = 2, 3 \bmod 4$, $\mathfrak{o} = \mathbb{Z}[\sqrt{-D}]$, and the co-area of $\mathfrak{o}$ is

$$\det \begin{pmatrix} \mathrm{Re}\,(1) & \mathrm{Im}\,(1) \\ \mathrm{Re}\,(\sqrt{-D}) & \mathrm{Im}\,(\sqrt{-D}) \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{D} \end{pmatrix} = \sqrt{D}$$

These conditions mod 4 also determine whether the conductor N is D, or $4D$, and in all cases

$$\text{co-area}(\mathfrak{o})^2 = \frac{1}{4} \cdot N \qquad\qquad \text{(in a naive normalization)}$$

(Recall the) **Claim:** The Gauss sum for a character of conductor N has absolute value $\sqrt{N}$.

Proof: Starting the computation in the obvious fashion, writing $\psi(a) = e^{2\pi i a/N}$. Let Σ' denote sum over $(\mathbb{Z}/N)^\times$, and Σ'' denote sum over $\mathbb{Z}/N - (\mathbb{Z}/N)^\times$.

$$\Big| \sum_{a \bmod N} \chi(a)\, \psi(a) \Big|^2 = \sum_{a,b}{}' \chi(a)\, \psi(a)\, \overline{\chi}(b)\, \psi(-b)$$

Replacing a by ab, this becomes

$$\sum_{a,b}{}' \chi(a)\,\psi\big((a-1)\cdot b\big)$$

We claim that, because χ has conductor N (and not smaller!)

$$\sum_{a}{}' \chi(a)\,\psi((a-1)\cdot b) \;=\; 0 \qquad\qquad (\text{for } \gcd(b,N)>1)$$

To see this, let p be a prime dividing $\gcd(b,N)$. That N is the conductor of χ is to say that χ is *primitive* mod N, meaning that χ does not factor through *any* quotient $\mathbb{Z}/(N/p)$. That is, there is some $\eta = 1 \bmod N/p$ such that $\chi(\eta) \neq 1$.

Since $p|b$, and $\eta = 1 \bmod N/p$,

$$(a\eta - 1)\cdot b \;=\; (a-1)b + a(\eta-1)b \;=\; (a-1)b \bmod N$$

Thus, replacing a by ηa,

$$\sum_a{}' \chi(a)\,\psi((a-1)\cdot b) \;=\; \sum_a{}' \chi(a\eta)\,\psi((a\eta-1)\cdot b)$$

$$=\; \chi(\eta)\sum_a{}' \chi(a)\,\psi((a-1)\cdot b)$$

Thus, the sum over a is 0. Thus, we can drop the coprimality constraint:

$$\sum_{a,b}{}' \chi(a)\,\psi\big((a-1)\cdot b\big) \;=\; \sum_{a,b} \chi(a)\,\psi\big((a-1)\cdot b\big)$$

For $a \neq 1$, the inner sum over b is 0, because the sum of a non-trivial character over a finite group is 0. For $a = 1$ the sum over b gives N. ///

Thus, the absolute value of the Gauss sum for *any* character with conductor exactly N is $\sqrt{N}$.

Returning to the class number formula for complex quadratic fields,

$$h(\mathfrak{o}) \;=\; \frac{\varepsilon\cdot|\mathfrak{o}^\times|}{2}\sum_{a \bmod N}\Big(\frac{a}{N}-\tfrac{1}{2}\Big)\cdot\chi(a) \qquad\qquad \text{(for some } |\varepsilon|=1)$$

The number of summands can be reduced by a factor of 2, as follows. Since $\chi(-1)=-1$, $\chi(N-a)=\chi(-a)=-\chi(a)$. Likewise,

$$\frac{N-a}{N}-\tfrac{1}{2} \;=\; 1-\frac{a}{N}-\tfrac{1}{2} \;=\; -\Big(\frac{a}{N}-\tfrac{1}{2}\Big)$$

Thus, we need only sum up over $a \;<\; N/2$. When $N/2$ is an integer, N was even, so divisible by 4, so $\chi(N/2)=0$. Thus,

$$h(\mathfrak{o}) \;=\; \varepsilon\cdot|\mathfrak{o}^\times|\sum_{1\le a<N/2}\Big(\frac{a}{N}-\tfrac{1}{2}\Big)\cdot\chi(a) \qquad\qquad \text{(for some } |\varepsilon|=1)$$

Example: $D = 3$ gives the Eisenstein integers $\mathfrak{o}$, which we know to have class number 1, since the ring is a PID. Here $|\mathfrak{o}^\times| = 6$.

$$|\mathfrak{o}^\times| \sum_{1 \le a < N/2} \big(\frac{a}{N} - \tfrac{1}{2}\big) \cdot \chi(a) \;=\; 6(\tfrac{1}{3} - \tfrac{1}{2}) \cdot (+1) \;=\; -1$$

Adjust by $\varepsilon = -1$ to obtain $h(\mathfrak{o}) = 1$, indeed.

Example: For $D = 5$, the conductor is $N = 20$ and $|\mathfrak{o}^\times| = 2$.

$$|\mathfrak{o}^\times| \sum_{1 \le a < N/2} \big(\frac{a}{N} - \tfrac{1}{2}\big) \cdot \chi(a)$$

$$= \; 2\Big((\tfrac{1}{20} - \tfrac{1}{2})(+1) + (\tfrac{3}{20} - \tfrac{1}{2})\big(\tfrac{-5}{3}\big)_2 + (\tfrac{7}{20} - \tfrac{1}{2})\big(\tfrac{-5}{7}\big)_2 + (\tfrac{9}{20} - \tfrac{1}{2})\big(\tfrac{-5}{9}\big)_2\Big)$$

$$= \; 2\Big((\tfrac{1}{20} - \tfrac{1}{2})(+1) + (\tfrac{3}{20} - \tfrac{1}{2})(+1) + (\tfrac{7}{20} - \tfrac{1}{2})(+1) + (\tfrac{9}{20} - \tfrac{1}{2})(+1)\Big)$$

$$= \; 2\Big(\tfrac{1}{20} + \tfrac{3}{20} + \tfrac{7}{20} + \tfrac{9}{20} - 2\Big) \;=\; -2$$

Adjust by $\varepsilon = -1$ to obtain $h(\mathfrak{o}) = 2$. This is not surprising, given

$$2 \cdot 3 \;=\; (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5})$$
